MASTER'S DEGREE THESIS

Master of Science in Advanced Mathematics and Mathematical Engineering

Title: The Hilbert transform: introduction and applications

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Abstract

The Hilbert transform is a linear operator applied to functions that doesn't change the domain of the function. It's widely used in signal processing thanks to the fact it defines a complex object, called analytic signal, that introduces the concept of instantaneous frequencies and Hilbert's spectrum, two useful tools in signal analysis. The Hilbert transform has several properties and, most important, it has also an important relation with the Fourier transform; this relation is the implementation key of the transform in any programming language. We will first introduce the Hilbert transform from the mathematical point of view, describing its properties and a discretization of the continuous case, in order to implement it in digital signals analysis. We will conclude introducing three of the most important applications the Hilbert transform has nowadays. Following this scheme then, in chapter 1 the formal introduction of the Hilbert transform will be given together with its basic properties and some examples of the explicit calculation; in chapter 2 we will study the relation with the Fourier transform while in chapter 3 the arguments for the discretization of the formulas are given with the purpose of implementing the theory in C++; going into applications we start with chapter 4, where the use of the analytic signal is given and used for the study of the electrocardiogram (ECG), while in chapter 5 we will achieve an efficient improvement of amplitude modulation techniques, and we will conclude in chapter 6 with the introduction of the Huang-Hilbert transform showing how it works and why is used although its problems. Most of the theorems and propositions used in this document are proved with proper proofs in the appendix A and B.

Contents

Chapter 1

The Hilbert transform

1.1 Introduction on the real line

To introduce the Hilbert transform just take a function $f(x)$ and call $H(f(x))$ the **Hilbert transform** of $f(x)$ where

$$H(f(x)) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(s)}{x-s} ds = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{|x-s| \geq \epsilon} \frac{f(s)}{x-s} ds \tag{1.1}$$

and PV means the Cauchy principal value, which is explicitly written in the second equality. The convergence of the integral is not obvious, and in most cases is not straightforward to calculate the Hilbert transform; while solving this integral one has to take care of the singularity at $s = x$. Anyway one maybe has to recall complex analysis tools in order to evaluate the integral depending on the function (like the residue theorem or Jordan's lemma). It has to be noticed also that, by the definition it follows, the Hilbert transform is just a convolution between the function $f(x)$ and the function $PV \frac{1}{\pi x}$. This will be useful later, once we introduce the relation with the Fourier transform. Despite the Fourier transform changes the domain of the function, an important result of the Hilbert transform is that it maintains the domain of $f(x)$ as it is.

1.2 Properties

We will introduce and prove now some properties of the Hilbert transform showing also how this properties can simplify the explicit calculus of the transformation of trigonometric functions. Let's call $z(t) = H(x(t))$, $z_1(t) = H(x_1(t))$, $z_2(t) = H(x_2(t))$, and a, a_1, a_2 some constants. Then the following holds:

$$\begin{aligned}
&\textbf{Linearity}: H(a_1x_1(t)+a_2x_2(t)) = a_1H(x_1(t))+a_2H(x_2(t)) && (1.2a)\\
&\textbf{Scaling}: H(x(at)) = z(at), a>0 && (1.2b)\\
&\textbf{Derivative}: H(x'(t)) = z'(t) && (1.2c)\\
&\textbf{Time-Shift}: H(x(t-a)) = z(t-a) && (1.2d)\\
&\textbf{Time reversal}: H(x(-at)) = -z(-at), a>0 && (1.2e)\\
&\text{If} f(t) \text{is } \textbf{even}, \text{ its Hilbert transform is } \textbf{odd.} && (1.2f)\\
&\text{If} f(t) \text{is } \textbf{odd}, \text{ its Hilbert transform is } \textbf{even.} && (1.2g)
\end{aligned}$$

In order to prove the property (1.2c) we need differentiability so we will prove it in the next chapter. Properties (1.2b),(1.2d) and (1.2e) are just a matter of change of coordinates, and property (1.2a) directly follows from the fact that by definition the Hilbert transform is a linear operation. To prove (1.2f) and (1.2g) consider

$$H(f(t)) = \frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{f(s)}{t-s}ds = \frac{1}{\pi}PV\int_0^{\infty}\left(\frac{f(s)}{t-s}+\frac{f(-s)}{t+s}\right)ds$$

then if $f(t)$ is even, that is $f(-t)=f(t)$, we get

$$\frac{1}{\pi}PV\int_0^{\infty}\left(\frac{f(s)}{t-s}+\frac{f(s)}{t+s}\right)ds = \frac{1}{\pi}PV\int_0^{\infty}\left(\frac{(t+s)f(s)+(t-s)f(s)}{t^2-s^2}\right)ds =$$

$$= \frac{2t}{\pi}PV\int_0^{\infty}\frac{f(s)}{t^2-s^2}ds$$

and if $f(t)$ is an odd function, that is $f(-t)=-f(t)$, it holds that

$$\frac{1}{\pi}PV\int_0^{\infty}\left(\frac{f(s)}{t-s}-\frac{f(s)}{t+s}\right)ds = \frac{1}{\pi}PV\int_0^{\infty}\left(\frac{(t+s)f(s)-(t-s)f(s)}{t^2-s^2}\right)ds =$$

$$= \frac{2}{\pi}PV\int_0^{\infty}\frac{sf(s)}{t^2-s^2}ds$$

□

1.3 Explicit calculations

1.3.1 Recalls

It's not always so easy to solve the integral defined by Hilbert transform. We should recall some tools to be used later on some examples:

Theorem 1.1(Cauchy's integral theorem):
if $f(z)$ is analytic in some simply connected region R then

$$\int_\gamma f(z)dz = 0 \tag{1.3}$$

where γ is a closed contour completely contained in R.

Theorem 1.2(Jordan's lemma):
If C_R^+ is the semicircle $z = Re^{i\phi}$, $\phi : 0 \to \pi$ in the upper half plane, $a > 0$ and $R > 0$, then

$$\int_{C_R^+} |e^{iaz}||dz| \leq \frac{\pi}{a} \tag{1.4}$$

Theorem 1.3:
Let $f(z)$ be analytic in some neighbourhood of z_0 where it has a simple pole and C_ϵ is a circular arc defined by $C_\epsilon : z = z_0 + \epsilon e^{i\phi}$ with $\phi : \phi_1 \to \phi_2$, then

$$\lim_{\epsilon\to 0^+} \int_{C_\epsilon} f(z)dz = i(\phi_2 - \phi_1)Res_{z=z_0} f(z) \tag{1.5}$$

1.3.2 Examples

Let's try now to evaluate some Hilbert transforms in easy cases. Let's start with the easiest case, a constant function $x(t) = c$. We can write its Hilbert transform explicitly:

$$H(c) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{c}{t-s} ds = \frac{c}{\pi} PV \int_{-\infty}^{\infty} \frac{1}{t-s} ds = 0 \tag{1.6}$$

But it happens to use the recalls we did in the past section in order to solve the integral defined by the transform. As an example let's try to evaluate the Hilbert transform $H(e^{i\omega t})$:

$$H(e^{i\omega t})(t) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{e^{i\omega s}}{t-s} ds \tag{1.7}$$

Start considering $\omega > 0$. Define $f(z) = \frac{1}{\pi}\frac{e^{i\omega z}}{t-z}$ and the contour $C = C_R^+ + L_{\epsilon,R}^2 + C_\epsilon + L_{e,R}^1$ which is closed and positively oriented. C_R^+ is a semicircle with radius R in the upper half-plane with the parametrization $z = Re^{i\phi}, \phi : \pi \to 0$. $L_{\epsilon,R}^2$ is the straight line from $z = t + \epsilon$ to $z = R$ on the real line. A sketch of the contour C is provided in figure 1.1

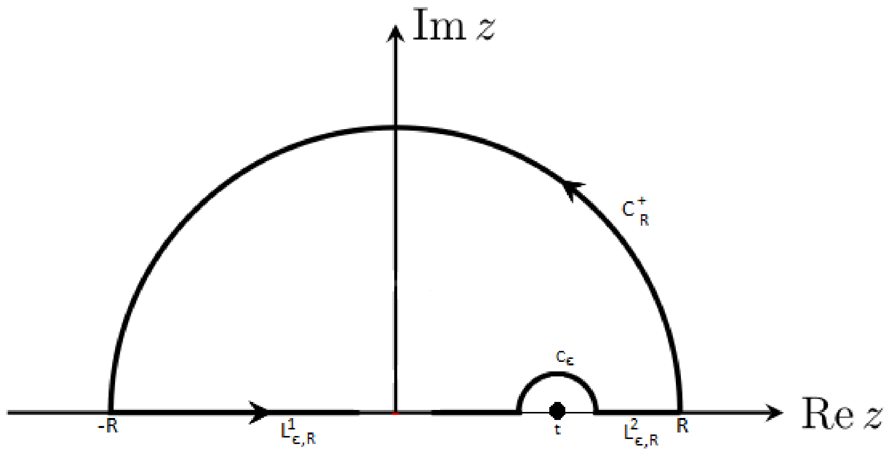

Figure 1.1: A sketch of the contour.

Since $f(z)$ is analytical both on and inside the contour C we have

$$\int_C f(z)dz = 0$$

because of Cauchy's integral theorem. Breaking up the integral thus yields

$$\int_{L^1_{\epsilon,R}+L^2_{\epsilon,R}} f(z)dz = -\int_{C_R^+} f(z)dz - \int_{C_\epsilon} f(z)dz \tag{1.8}$$

because $\omega > 0$ and C_R^+ is in the upper half-plane Jordan's lemma gives us

$$|\int_{C_R^+} f(z)dz| \leq \int_{C_R^+} |f(z)||dz| = \int_{C_R^+} \frac{|e^{i\omega z}|}{|\pi(t-z)|}|dz| \leq \frac{1}{\pi}\frac{1}{|R-t|}\int_{C_R^+} |e^{i\omega z}||dz| \leq$$

$$\leq \frac{1}{\pi}\frac{1}{|R-t|}\frac{\pi}{2} = \frac{1}{2}\frac{1}{|R-t|}.$$

Furthermore, for the integral over the contour C_ϵ, we get by Theorem 1.3

$$\int_{C_\epsilon} f(z)dz = i(0-\pi)Res_{z=t}f(z) = -i\pi\frac{e^{i\omega t}}{-\pi} = ie^{i\omega t}.$$

Of course, the integral over the contour $L^1_{\epsilon,R} + L^2_{\epsilon,R}$ in (1.7) coincide with the integral (A.1 in appendix A), when $R \to \infty$ and $\epsilon \to 0^+$. For a fixed t, applying said limits and $\omega > 0$ in (1.8) yields

$$\frac{1}{\pi}PV\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{t-s}ds = -0 - ie^{i\omega t} = -ie^{i\omega t}, \omega > 0$$

and for the case $\omega < 0$ similar arguments give us

$$\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{t-s}ds = ie^{i\omega t}, \omega < 0$$

while for $\omega = 0$ the complex exponential function is a constant and this is the previous case. So we can conclude that for all $\omega \in \mathbb{R}$

$$H(e^{i\omega t}) = -i\mathrm{sgn}(\omega)e^{i\omega t}. \tag{1.9}$$

But it's not even always so hard to evaluate this transform. For example in some cases one can use the properties (1.2a)...(1.2g) we introduced above to solve the Hilbert transform of trigonometric functions. For example, supposing $\omega > 0$:

$$\begin{aligned} H(\cos(\omega t) = H\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right) = \frac{H(e^{i\omega t})+H(e^{i(-\omega)t})}{2} = \\ = \frac{-i\mathrm{sgn}(\omega)e^{i\omega t} - i\mathrm{sgn}(-\omega)e^{i(-\omega)t}}{2} = \\ = \frac{-ie^{i\omega t}+ie^{-i\omega t}}{2} = \frac{e^{i\omega t}-e^{-i\omega t}}{2i} = \sin(\omega t) \\ H(\sin(\omega t)) = H(\cos(\omega t - \frac{\pi}{2})) = \sin(\omega t - \frac{\pi}{2}) = -\cos(\omega t) \end{aligned}$$

but it's also true that if $\omega < 0$ then $H(\cos(\omega t)) = -\sin(\omega t)$, and $H(\sin(\omega t)) = -(-\cos(\omega t)) = \cos(\omega t)$ by the time reversal property and then we can just write it in a compact form for all $\omega \in \mathbb{R}$:

$$H(\cos(\omega t)) = \mathrm{sgn}(\omega)\sin(\omega t) \tag{1.10}$$

$$H(\sin(\omega t)) = -\mathrm{sgn}(\omega)\cos(\omega t) \tag{1.11}$$

Chapter 2

Relation with the Fourier transform

2.1 The Fourier transform

We want to recall briefly some informations about the Fourier transform, which will be used later. We start by defining a function $f \in S(\mathbb{R})$ with $f : \mathbb{R} \to \mathbb{C}$, where $S(\mathbb{R})$ is the Schwartz space, that is the set of all indefinitely differentiable functions f so that f and all its derivatives $f', f'', ..., f^{(l)}, ...,$ are **rapidly decreasing** in the sense that

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty, \text{ for every } k, l \geq 0;$$

then we define the **Fourier transform** of f as

$$F(f(x)) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx, \xi \in \mathbb{R}$$

And we define also the **Inverse Fourier transform** as:

$$F^{-1}(\hat{f}(\xi)) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi$$

where $\xi = 2\pi k$. Notice that we have different notations and conventions available to define the equations above; what is important is that if we combine the two transformations (direct and inverse) together then it has to result in a factor of $\frac{1}{2\pi}$ in front of the integral and an opposite sign in the exponentials. Also this factor is explicit only if in the exponential we have the angular velocity ξ, if we have expressed directly the term $2\pi k$ then we can omit the factor in front of the integral. So we will indicate $\hat{f}(k)$ the definition without $\frac{1}{\sqrt{2\pi}}$ and $\hat{f}(\xi)$ otherwise. Some basic properties are:

(i) $f(x+h) \xrightarrow{\text{F}} \hat{f}(k)e^{2\pi ihk}$ *whenever* $h \in \mathbb{R}$.
(ii) $f(x)e^{-2\pi ixh} \xrightarrow{\text{F}} \hat{f}(k+h)$ *whenever* $h \in \mathbb{R}$.
(iii) $f(\delta x) \xrightarrow{\text{F}} \delta^{-1}\hat{f}(\delta^{-1}k)$ *whenever* $\delta > 0$.
(iv) $f'(x) \xrightarrow{\text{F}} 2\pi ik\hat{f}(k)$.
(v) $-2\pi ixf(x) \xrightarrow{\text{F}} \frac{d}{dk}\hat{f}(k)$.

The inversion formula and the basic properties of the Fourier transform are proved in Appendix B. An important result about the Fourier transform is the convolution theorem. Indicating with $*$ the convolution between two functions, assuming $f, g \in S(\mathbb{R})$ we have (using $z = x - y$)

$$F(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)dy \right) dx =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(y) \left(\int_{-\infty}^{\infty} f(x-y)e^{-i\xi x}dx \right) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(y) \left(\int_{-\infty}^{\infty} f(z)e^{-i\xi(z+y)}dz \right) dy =$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)e^{-i\xi y}dy \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)e^{-i\xi z}dz \right) =$$

$$= F(f(x))F(g(x))$$

We have that the Fourier transform of a convolution is the product of the Fourier transforms. We want now to expand this definition to the spaces L^1 and L^2. Let's define C_0 as the set of continuous functions that tend to zero at infinity. Since $S(\mathbb{R})$ is the set of functions rapidly decreasing, and since each function on S is uniformly continuous (by the fact the function itself and all its derivatives are rapidly decreasing) then $S(\mathbb{R}) \subset C_0$. Take $\phi \in S(\mathbb{R})$ then

$$\sqrt{(2\pi)}|F(\phi(k))| = |\int e^{-ikx}\phi(x)dx| \leq \int |\phi(x)|dx$$

and taking the supremum of this inequality over k we get

$$\sqrt{(2\pi)}||F(\phi)||_\infty \leq ||\phi||_1$$

Since the Schwartz space S in dense in L^1 (see [4] pag. 309), taking $f \in L^1$, there exist a sequence $(\phi_m) \in S(\mathbb{R})$ that converges to f in the L^1-norm. Given the fact that if a function is in the Schwartz space then its Fourier transform is in the same space, then $F(\phi_m)$ is a sequence for which

$$\sqrt{(2\pi)}||F(\phi_m) - F(\phi_l)||_\infty \leq ||\phi_m - \phi_l||_1$$

is true and it converges to some Fourier transform of some function $F(g(x)) \in C_0$ and also it's true that

$$\sqrt{(2\pi)}|F(g(x)) - F(f(x))| = \sqrt{(2\pi)} \lim_{m\to\infty} | \int [\phi_m(x) - f(x)]e^{-ikx}dx| \leq$$

$$\leq \lim_{m\to\infty} ||\phi_m - f||_1 = 0$$

and this proves that actually $F(g(x)) = F(f(x))$ and we have well defined the Fourier transform in L^1 as a linear map from L^1 to C_0(see [4] pag. 309). Now we will just comment the inversion theorem in L^1. We need to assume that f is a function in L^1 whose Fourier transform is also in L^1, then the following integral is well defined

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(t)e^{ixt}dt$$

Defining $H(t) = e^{-|t|}$ and

$$h_\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\lambda t)e^{itx}dt$$

Then by Fubini theorem

$$(f * h_\lambda)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\lambda t)F(f(x))e^{ixt}dt$$

Now as λ goes to 0 then $H(\lambda t)$ goes to 1 and the right part of the last equality goes to $g(x)$. Also defining a sequence $\lambda_n \to 0$ as $n \to \infty$ it can be proven that

$$\lim_{n\to\infty} (f * h_{\lambda_n})(x) = f(x) \quad a.e.$$

and then $f(x) = g(x)$ almost everywhere (see [10] pag. 185), which is what we wanted to show. Since not every function belonging to L^1 belongs also to L^2 then we can't use the above definition of Fourier transform for functions exclusively in L^2. Take $\phi, \psi \in S$ then (noticing $\overline{F(\phi)} = F^{-1}(\overline{\phi})$) we can write the inner product in the L^2 sense as

$$< F(\phi), F(\psi) >= \int \overline{F(\phi)}F(\psi)dx =< \phi, \psi >$$

Then the Fourier transform is an isometric map from $S \subset L^2$ to $S \subset L^2$. Also S is dense in L^2 (see [4] pag. 311) then by this fact and the previous one we can extend the Fourier transform to the whole L^2 (by the fact that a bounded linear transformation can be extended to the completion of its domain). We also need to prove Plancherel's theorem which states: given $f, g \in L^2$

$$< F(f), F(g) >= \int \overline{F(f)}F(g) =< f, g >$$

and in particular

$$\int |f(x)|^2 dx = \int |F(f)(k)|^2 dk.$$

Then choosing a sequence ϕ_n in $S(\mathbb{R})$ that converges to f in L^2 we have that $F(f)(k)$ is the limit in the L^2-sense of $F(\phi_n)$.

$$F(f)(k) = \lim_{R\to\infty} \int_{|x|\leq R} f(x)e^{-ikx} dx$$

The inverse transform may be computed in a similar way (see [4] pag 312).

2.2 Hilbert transform through Fourier transform definition

Since, as we have seen, the Hilbert transform is actually a convolution between two functions, it's obvious starting to think about the Fourier transform. We want to show then how this two transformations are related.

Theorem 2.0 (Riesz theorem) : *the Hilbert transform is a bounded linear operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for $1 < p < \infty$.*

We will not prove it here because it requires interpolation of operators, but we can find an easy proof (see [5] pag. 24) for the case $p = 2$. Also

Theorem 2.1: *Suppose $x(t) \in L^p(\mathbb{R})$, $1 < p \leq 2$ and $F(x(t))$ is the Fourier transform of $x(t)$. Then the Fourier transform of $H(x(t))$ is given by*

$$F[H(x(t))] = (-isgn(k)) \cdot F(k) \tag{2.1}$$

Where k represents the frequency variable.
Notice that this theorems do not hold for $p = 1$ because $x(t) \in L^1(\mathbb{R})$ doesn't always imply that $H(x(t)) \in L^1(\mathbb{R})$. Take for example $x(t) = \chi_{[0,1]}(t) \in L^1(\mathbb{R})$. Then the Hilbert transform is

$$H(x(t)) = \frac{1}{\pi} PV \int_0^1 \frac{1}{t-s} ds = \frac{1}{\pi} \log |\frac{t}{t-1}|$$

and clearly $H(x(t)) \notin L^1(\mathbb{R})$. The Fourier transform of this Hilbert transform does not exist in the usual sense so the theorem above does not hold. But anyway it is true that if both $x(t), H(x(t)) \in L^1(\mathbb{R})$ then the theorem holds also for $p = 1$. For $p = 2$ observe that taking the norm of (2.1) with $x(t) \in L^2$ and using Plancherel's theorem we get $||H(x(t))||_2 = ||x(t)||_2$ proving that the Hilbert transform is not only a bounded operator in L^2, it is in fact an isometry.

2.3 Differentiation and inversion

Once we have related Hilbert and Fourier transforms we can introduce other properties deriving directly from Fourier analysis. First of all recall property (1.2c) $H(x'(t)) = z'(t)$ with $z(t) = H(x(t))$; since we didn't prove it yet, let's do it now:

Theorem 2.2 (Differentiation) : If $x(t) \in L^p(\mathbb{R})$, $1 < p \leq 2$ is differentiable, then:

$$H(x^{'}(t)) = \frac{d}{dt}H(x(t))$$

To prove this, one has just to remember differentiability property of Fourier transform and write:

$$F(H(x^{'}(t))) = (-i\text{sgn}(k) \cdot F(x^{'}(t)) = (-i\text{sgn}(k)) \cdot (2\pi ik \cdot F(x(t))) =$$

$$= 2\pi ik \cdot (-i\text{sgn}(k)) \cdot F(x(t)) =$$

$$= 2\pi ik \cdot F(H(x(t))) = F(\frac{d}{dt}H(x(t))$$

One may think also what happens if we apply twice the Hilbert transform

Theorem 2.3 (Inversion) : If $x(t) \in L^p(\mathbb{R})$, $1 < p \leq 2$. Then

$$H(H(x(t))) = -x(t)$$

To prove it just consider

$$F(H(H(x(t)))) = (-i\text{sgn}(k)) \cdot F(H(x(t))) =$$

$$= (-i\text{sgn}(k)) \cdot (-i\text{sgn}(k)) \cdot F(x(t)) =$$

$$= i^2 \cdot (\text{sgn}(k))^2 \cdot F(x(t)) = -F(x(t)) = F(-x(t))$$

Then it's true that $H^{-1} = -H$.

Chapter 3

Discretization and implementation

Once we introduced all the theoretical background we want to develop and study some applications. To do this we need to introduce also the discretization of a continuous problem such as the Hilbert transform. As we said before we can use the relation with the Fourier transform to introduce the digital and discrete implementation of the Hilbert transform. Recall that when discretizing continuous formulas one has to introduce the **sampling frequency** which is the frequency we use to take discrete data from the signal, so the time interval between two consecutive samples. An important theorem about the sampling frequency is that if we want to avoid errors in approximations and other problems like **aliasing** we need to have a sampling frequency at least the double of the maximum frequency contained in the signal (Nyquist's theorem). Now we have to think the digitalized signal as a sequence of N samples, each of them containing the value of the signal at a specific time. Let's introduce now the **Discrete Fourier Transform (DFT)** which is the one used in digital analysis (actually the most used is the optimized algorithm of the **Fast Fourier Transform (FFT)** which is not our purpose in this presentation). The **DFT** is also a perfect example of how much intuitive is translating continuous problems into discrete ones. We define the **DFT** and it's inverse **IDFT** or **DFT**$^{-1}$ as (see [6] pag. 22):

$$F[k] = \sum_{n=0}^{N-1} x[n] e^{-i\frac{2\pi}{N}kn}, k = 0, 1, ..., N-1$$

and

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k] e^{i\frac{2\pi}{N}kn}, n = 0, 1, ..., N-1$$

Notice that this time k is a discrete variable and not continuous as before. Now we know that the Hilbert transform can be written as

$$H(x(t)) = F^{-1}[-i\text{sgn}(k)F(x(t))]$$

And that the Fourier transform of the convolution of two functions is the product of the Fourier transforms of each function. In the frequency domain we know that the Fourier transform gives us frequencies starting from $-\frac{N}{2}$ up to $\frac{N}{2}$. This means that assuming N even and starting counting from 0 the signum function we used above has value 1 for $[k = 1, 2..\frac{N}{2} - 1]$, 0 for $[k = 0, \frac{N}{2}]$ and then -1 for $[k = \frac{N}{2} + 1, ...N - 1]$. For N odd is just a matter of rewriting the three intervals as $[k = 1, 2..\frac{N-1}{2}]$,$[k = 0, N]$ and $[k = \frac{N+1}{2}, ...N - 1]$ respectively. Considering so all the cases we can then write a unique formula for the **Discrete Hilbert Transform (DHT)** as :

$$H_d(x(t)) = H(x_i) = DFT^{-1}[-iS[k]F[k]]$$

where $S[k] = \text{sgn}(\frac{N}{2} - k)\text{sgn}(k)$ which satisfies all the above cases, for N even or odd. So, whatever the purpose of the implementation is, the algorithm to evaluate the **DHT** is:

1) Given $x(t)$ extrapolate N samples x_i.
2) Calculate the **DFT** of the discrete signal.
3) Multiply the **DFT** by the vector $-iS[k]$.
4) Calculate **IDFT** of what we have obtained

This is the basic idea in all the following implementations in C++.

Chapter 4

Electrocardiogram (ECG)

Among all the tools that medicine has nowadays one of the most important is electrocardiogram. Abbreviated as ECG this is a record of the electrical activity in a person's heart over time. Being able to analyze electrical signals from an heart is useful to determine several diseases and irregularities; as example we can just think about irregularities in time intervals between an heartbeat and a following one (arrhythmia). We will focus our attention on the most important part of an ECG which is the heartbeat itself. Physiological signals (as ECG) are considered to be quasi periodic (read introduction in [8]). They are non stationary then a Fourier series is not efficient for ECG. The heartbeat is technically called **QRS-Complex** which is that part of the electric heart's wave that corresponds to the depolarization of the right and left ventricles. A too large or too short QRS-Complex means problems in electrical conductivity so that's why it's important to extract and analyze it.

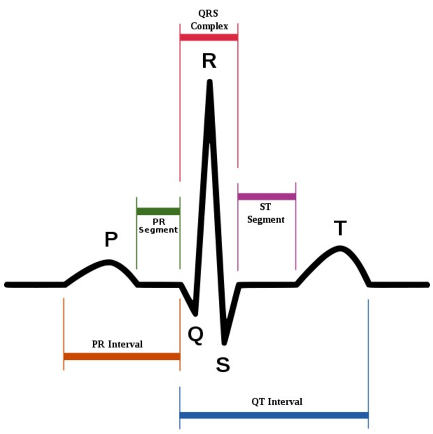

Figure 4.1: *An heartbeat. QRS-Complex is the most distinctive part of the single beat.*

As we can see, the QRS-Complex is a sort of deformed sine wave. And we will see that this is not just a forced and imaginary relation between the two waves. For the moment we want to recall that a sine wave $\sin(t)$ has an Hilbert transform of $H(\sin(t)) = -\cos(t)$. Let's build now what is called an **analytic signal**: a complex function whose real part is the original and real signal and the imaginary part is its Hilbert transform. In general an analytic signal is built from a real signal $x(t)$ and it's defined as

$$z(t) = x(t) + iH(f(t))$$

and in the case of the sine function it's

$$z(t) = \sin(t) - i\cos(t) = -ie^{it}$$

This analytic signal has as plot, in this case, the unit circle in the complex plane, centred in zero.

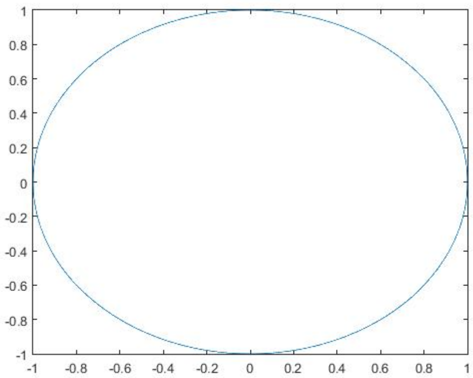

Figure 4.2: *Plot of a the analytic signal defined above.*

We will see in a while that this plot will be similar to the one we obtain in the real case using a real ECG signal. Since we have to deal with real physiological signals of a human's heart we will use a well known on-line database to download our data [2]: *Physionet.org*. Let's see how does an ECG looks like , call it *ECG(t)*, take a five second record and plot it. In Figure 4.3 we can recognize five different QRS-Complex peaks with the corresponding P and T waves. It's clear now that it is actually a periodic signal. To analyze the peaks corresponding to the QRS-Complex the signal must be noiseless and it's necessary to have a noticeable difference between the maximum of QRS-Complex and the maximum of P or T waves, otherwise every automatic analyzing algorithm will fail in taking exclusively the QRS-Complex peaks.

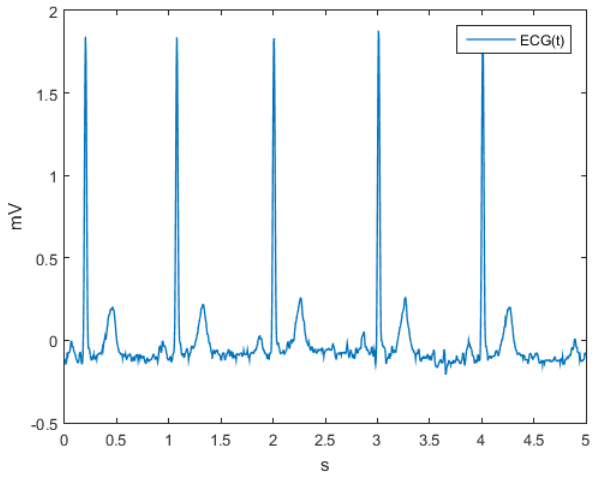

Figure 4.3: *Five second record of a human heart. We recognize easily all the parts indicated in Figure 4.1.*

Now it comes its analysis. To see what an Hilbert transform of an ECG looks like, call it $H(ECG(t))$ overlap it over the original ECG signals, we obtain

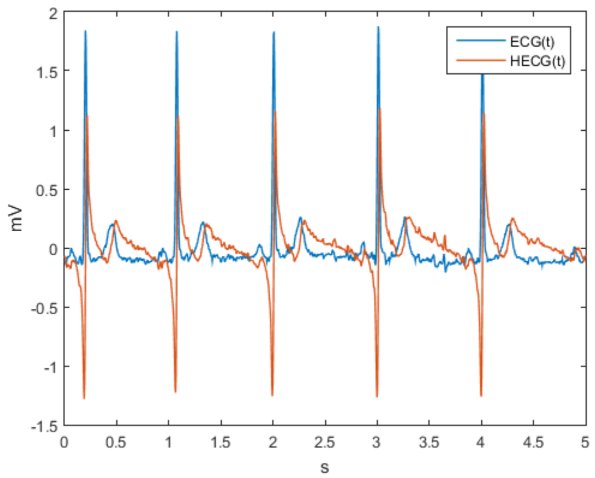

Figure 4.4: *The Hilbert transform (orange) and the ECG (blue) overlapped.*

As we can intuitively see, the behaves of the Hilbert transform outside the peaks' range is like $1/t$, this is more clear if we filter the signal and we cut off all the parts of the ECG except for the QRS-Complex. Let's build now the analytic signal from $ECG(t)$, $z(t) = ECG(t) + iH(ECG(t))$ and plot it in the complex plane (Figure 4.5).

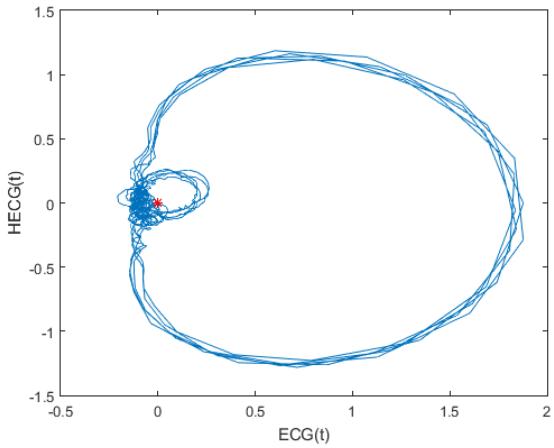

Figure 4.5: *Plot of z(t).*

First of all we deduce there are two different types of closed curves. We have five big closed curves corresponding to the QRS-Complex of each heartbeat and five more small closed curves given by the rest of the signal oscillating around the origin. As we can see now it comes back the relation with the unit circle, proving the fact that a QRS-Complex is actually a sort of deformed sine wave, this will be useful to confirm our results once we have implemented and tested the theory in C++. What has to be highlighted is the fact that if we filter the signal and rebuild the analytic signal, the plot will be without the five smaller closed curves around the origin, becoming even more similar to the parametrization of the unit circle. In fact take only the first second of $ECG(t)$ (only the first peak)and call it $QRS(t)$, filter it, and plot its analytic expansion in the complex plain. What we obtain is exactly what we expected:

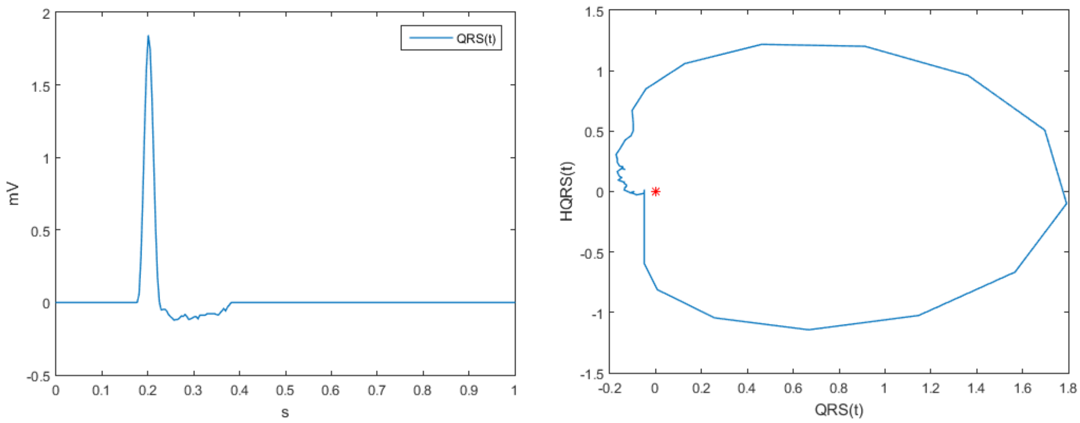

Figure 4.6: *Plot of filtered QRS(t) (on the left) and its analytic expansion (on the right).*

4.1 C++ implementation results

As discussed in Chapter 3, the algorithm to implement the theory of the Hilbert transform in C++ is just a matter of evaluating some Fourier transforms (both direct and inverse). In the Appendix C one can find the important parts of the code used to produce the results that are going to be presented here. Keep in mind the code is wrote in order to be easy for the reader to understand it, so it's not optimized at all. From the database we have introduced above, take an ECG wich is longer than the previous one

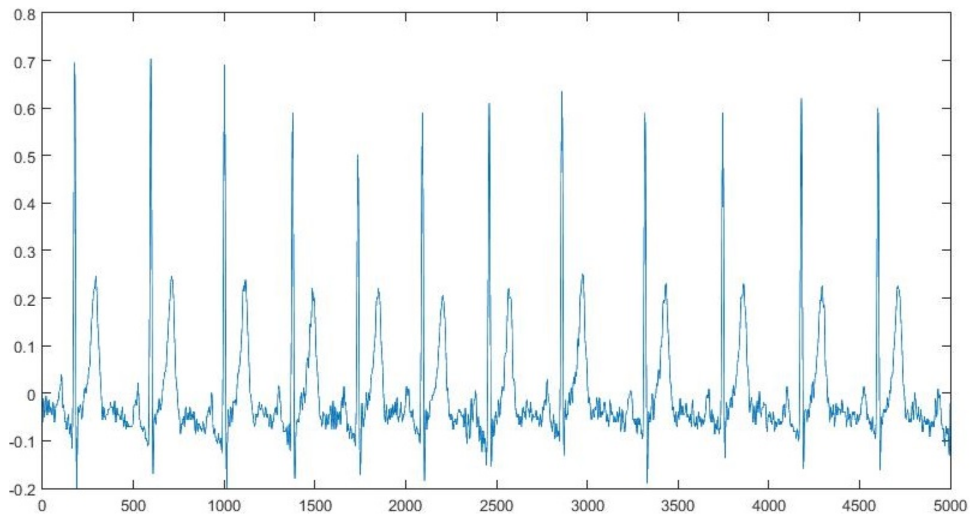

Figure 4.7: *ECG(t) in mV over 5000 samples. Notice that the sampling rate is 2 milliseconds so the total time is 10 seconds.*

We can clearly see that in this case the QRS-Complex part of each beat is much higher than the rest of the signal, then it's not a problem to filter the signal. The implementation of the filter is just a function which finds all maximum in the signal, eliminate the ones that are below a certain limit (what we consider the right level to separate each peak from the rest of the signal), and open a window around each maximum left to take the whole QRS-Complex. We want to show in the following figures the different results obtained using an unfiltered and a filtered signal. Notice how easy is to analyze the signal once translated in its analytic form and plotted in the complex plane; to check irregularities in heartbeat one has just to count the number of closed curves and the distance between them.

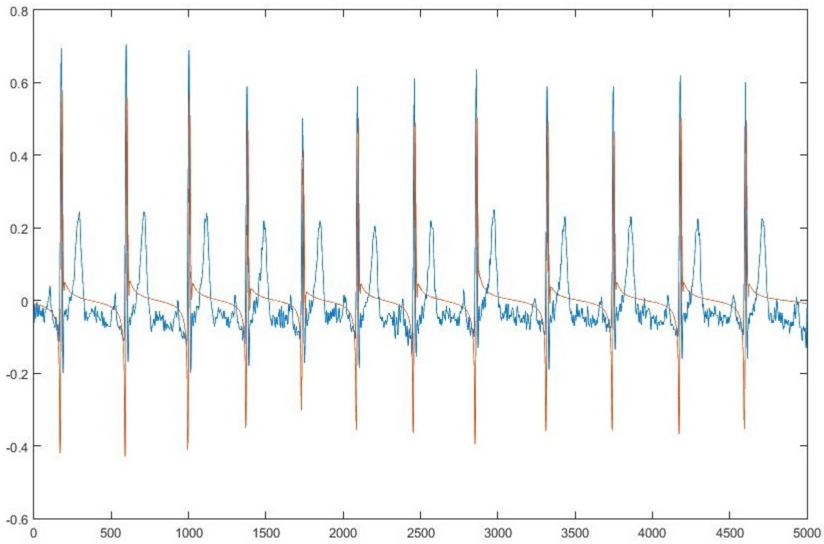

Figure 4.8: *The Hilbert transform (orange) overlapped to the unfiltered signal.*

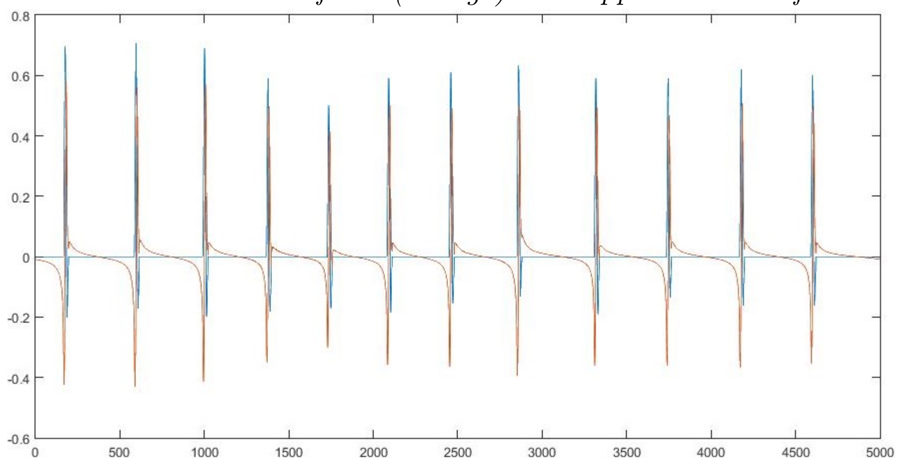

Figure 4.9: *The Hilbert transform (orange) overlapped to the filtered signal.*

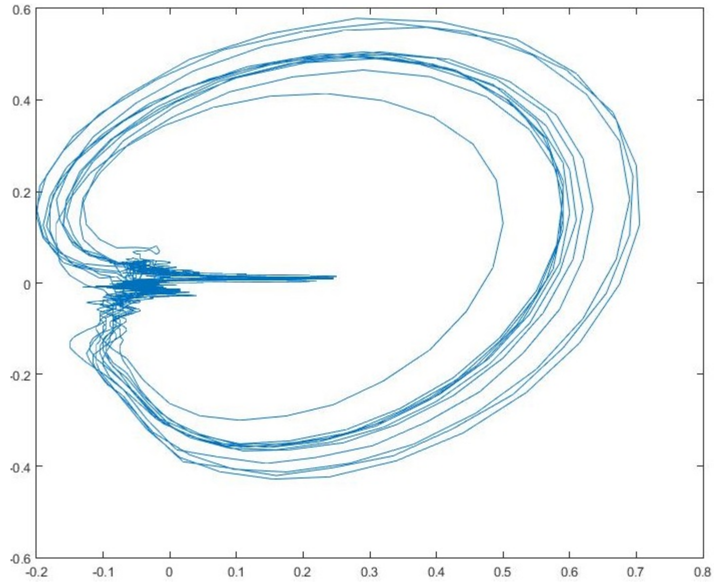

Figure 4.10: *The Hilbert transform over the unfiltered signal.*

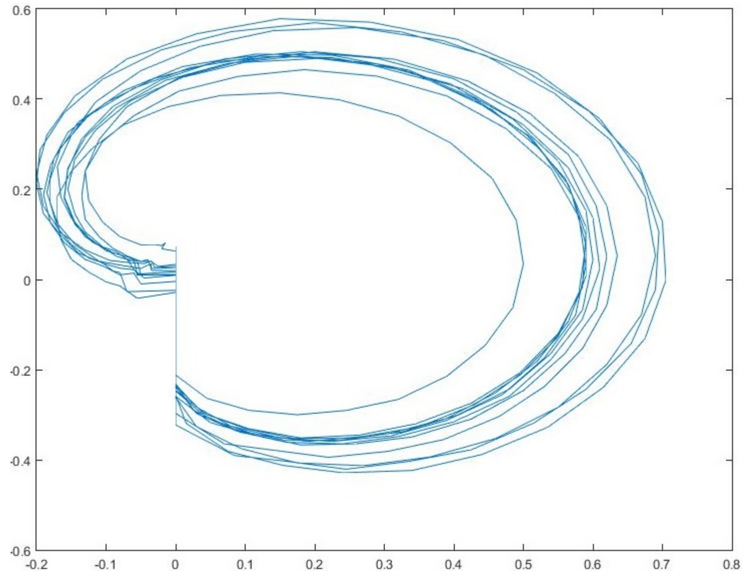

Figure 4.11: *The Hilbert transform over the filtered signal.*

4.2 Comments

Talking about the limits of this method we have to say that the only case in which we can't apply this theory is when one or more QRS-Complex are not distinguishably higher from the rest of the signal; in this case is not obvious to determine what has to be analyzed and what not. This method is not used only to analyze the ECG in order to find some irregularities but it's also used to study heart's responses to medications and therapies. The analytic signal built from a real signal is also an important concept used nowadays in signal analysis as we are going to see in the next applications.

Chapter 5

Amplitude modulation (AM)

The amplitude modulation is a technique used expecially in radio communications; even if today the frequency modulation is the one which is used more, the amplitude modulation was the first technique used and studied, which also gave us the theory to develope other techniques. Modulation is a process in which a signal is modified by a so called carrier wave, which has a higher frequency. The purpose is to move the frequency spectrum of the modulated signal at higher or lower frequencies depending on what we want to obtain and this is done, in this case, by changing the amplitude of the carrier wave. Take a signal $f(t)$ and its Fourier transform $F(k)$ where k is the frequency variable. Suppose now $F(k) = 0$ if $|k| > C$ for some C (the Fourier transform is bounded) and suppose also $F(0) = 0$. An **amplitude modulation** could be

$$f_{AM}(t) = f(t)\cos(2\pi k_c t) \tag{5.1}$$

where k_c is called the *carrier frequency* and is much larger than the bandwidth of $f(t)$, that is, $k_c >> C$. Then the Fourier transform could be

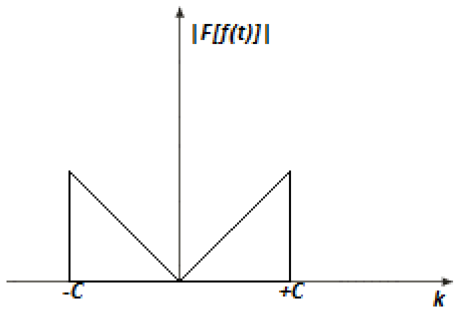

Figure 5.1: *The Fourier transform of the signal (not modulated), that is the amplitude spectrum of the Fourier transform. Notice that since the signal is a purely real signal there is a symmetry with respect to the origin.*

On the other hand if we calculate the Fourier transform of the modulated signal this is

$$F(f_{AM}(t)) = F(f(t)\cos(2\pi k_c t)) = F(f(t)) * F(\cos(2\pi k_c t)) =$$
$$= F(f(t)\frac{1}{2}(\delta(k-k_c)+\delta(k+k_c)) = \frac{1}{2}(F(k-k_c)+F(k+k_c))$$

which has the following plot

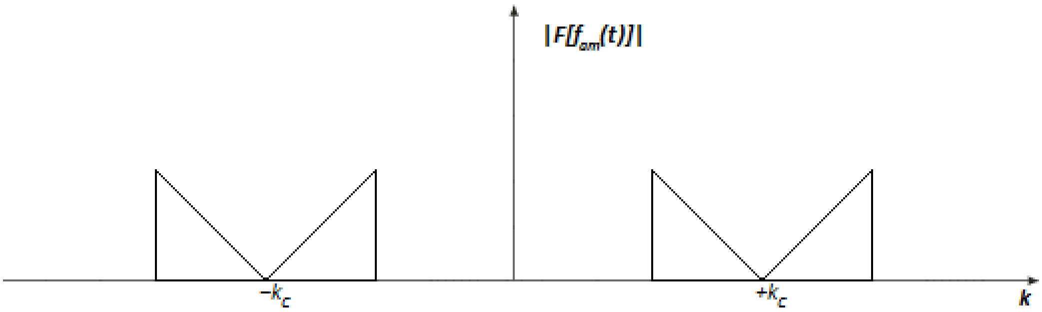

Figure 5.2: *As we can notice, the effect of the modulation is to translate the amplitude spectrum of the signal around $-k_c$ and $+k_c$.*

We can also recover the signal. This process is what is called **demodulation**. If we are dealing with a digital modulation (meaning we can manipulate the digital signal with a computer) knowing the carrier frequency we can write

$$f_{AM}(t)\cos(2\pi k_c t) = f(t)\cos^2(2\pi k_c t) = f(t)\frac{1}{2}(\cos(4\pi k_c t)+1) =$$
$$= \frac{1}{2}f(t) + \frac{1}{2}f(t)\cos(4\pi k_c t) \tag{5.2}$$

and we can just invert the formula to obtain our original signal without any error. If, on the other hand, we are dealing with analogic signals and we want to demodulate without the use of a computer we need to use a low-pass filter: we use the last equality in the previous equation, remove the high frequencies k_c with a low-pass filter, and multiply by 2 the filtered signal. Up to here everything seems fine but we can still improve this method. We can clearly see that the bandwidth is doubled symmetrically to the origin, and this is a common problem in radio transmission because one wants to optimize the bandwidth in order to admit several signals in a frequency spectrum without any overlapping of signals, so we want to separate them in an efficient way. We want to eliminate this useless redundancy of information by defining what is called the **Single-sideband modulation (SSB)** which will eliminate half of the bandwith using the Hilbert transform and the definition of analytic signal.

5.1 SSB-modulation

We first need a theorem: Let $z(t) = f(t) + i\mathcal{H}(f(t))$ be an analytic signal with $f(t)$ a purely real signal. Then its Fourier transform, $F(z(t))$, is given by

$$F(z(t)) = (1 + \text{sgn}(k)) \cdot F(f(t)) = \begin{cases} 2F(f(t)) \ , & k > 0 \\ F(f(t))(0) \ , & k = 0 \\ 0, & k < 0 \end{cases}$$

$$F(z(t)) = F(f(t)) + F(iH(f(t))) = F(f(t)) + i \cdot F(H(f(t))) =$$

$$= F(f(t)) + i \cdot [-i\text{sgn}(k)) \cdot F(f(t))] = (1 + \text{sgn}(k)) \cdot F(f(t))$$

and the proof is complete. Note that for the sign function we have $\text{sgn}(0) = 0$. Now let's introduce the analytic expansion of our signal

$$z(t) = f(t) + iH(f(t))$$

Then by the previous theorem its Fourier transform is

$$F(z(t)) = \begin{cases} 2F(f(t)) \ , & k > 0 \\ F(f(t))(0) \ , & k = 0 \\ 0, & k < 0 \end{cases} = \begin{cases} 2F(f(t)) \ , & k > 0 \\ 0, & k \leq 0 \end{cases}$$

The fact that from now on the bandwidth will be the half of the previous case is due to the exhibit of Hermitian symmetry of $F(z(t))$, that is $F(z(t)(-k) = F(z(t))(k)$ and then $|F(z(t)(-k)| = |F(z(t))(k)|$. Now we want to modulate this analytic signal and extrapolate its real modulated part which will be the actual modulated real signal. Then

$$Z(t) = z(t)e^{2\pi i k_c t}$$

and its Fourier transform is

$$F(Z(t)) = F(z(t)e^{2\pi i k_c t}) = F(z(t)) * F(e^{2\pi i k_c t}) =$$

$$= F(z(t))\delta(k - k_c) = F(z(t))(k - k_c) = \begin{cases} 2F(f(t))(k - k_c) \ , & k > k_c \\ 0, & k \leq k_c \end{cases}$$

We want to recover now the real part of this signal which is supposed to be the real single-sideband modulated signal. $Z(t)$ is also an analytic signal since every analytic signal has only positive frequency components (due to its complexity). So call the real part $f_{\text{ssb}}(t)$ where ssb stands for *single-sideband*. We have

$$Z(t) = f_{\text{ssb}}(t) + iH(f_{ssb}(t)), Z(t) = z(t)e^{2\pi i k_c t}$$

Then we know that

$$f_{\text{ssb}}(t) = f(t) \cdot \cos(2\pi k_c t) - H(f(t)) \cdot \sin(2\pi k_c t) \tag{5.3}$$

And finally we evaluate the Fourier transform of $f_{\text{ssb}}(t)$:

$$F(f_{ssb}(t)) = F(f(t) \cdot \cos(2\pi k_c t) - H(f(t)) \cdot \sin(2\pi k_c t)) =$$

$$= F(f(t)) * F(\cos(2\pi k_c t)) - F(H(f(t))) * F(\sin(2\pi k_c t)) =$$

$$= F(f(t)) * \frac{1}{2}(\delta(k-k_c) + \delta(k+k_c)) - (-i\text{sgn}(k))F(f(t)) * \frac{1}{2i}(\delta(k-k_c) - \delta(k+k_c))$$

$$= \frac{1}{2}(1 + \text{sgn}(k-k_c))F(f(t))(k-k_c) + \frac{1}{2}(1 - \text{sgn}(k+k_c))F(f(t))(k+k_c) =$$

$$= \begin{cases} F(f(t))(k-k_c)\ , & k > k_c \\ 0, & -k_c \leq k \leq k_c \\ F(f(t))(k+k_c)\ , & k < -k_c \end{cases}$$

Which has the plot we wanted (halved bandwidth)

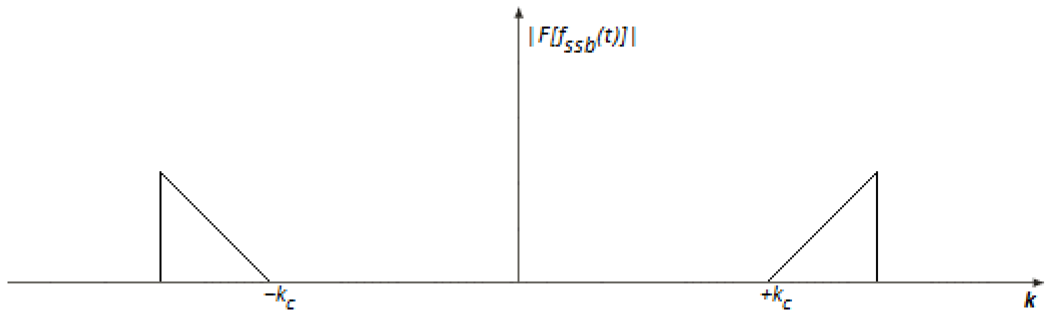

Figure 5.3: *Plot of* $F(f_{\text{ssb}}(t))$.

To be precise this method brought us to have the upper half band of the amplitude spectrum, if we want to obtain the lower half band we should consider $\overline{z(t)}$ and do the same process. The demodulation works as before and skipping calculations we have

$$f_{\text{ssb}}(t)\cos(2\pi k_c t) = \frac{1}{2}f(t) + \frac{1}{2}f(t)\cos(4\pi k_c t) - \frac{1}{2}f(t)\sin(4\pi k_c t) \tag{5.4}$$

And, as before, we should use a low pass filter and multiplicate by 2 if we are dealing with analogic signals or just invert the formula if we can manipulate it digitally.

5.2 C++ implementation results

We want to implement now both modulations in order to do a comparison. Since we are implementing in C++ is supposed to be a digital analysis so the doubled-sideband modulation/demodulation is given by the explicit formula (5.1)/(5.2) while the single-sideband modulation/demodulation is given by (5.3)/(5.4). Let's first use a simple signal: a sine function of frequency $3Hz$, $f(t) = \sin(2\pi 3t)$ modulated with a carrier frequency of $300Hz$. Then the x-axis represents the sample number k which refers to instant time $t = 0.001 \cdot k$.

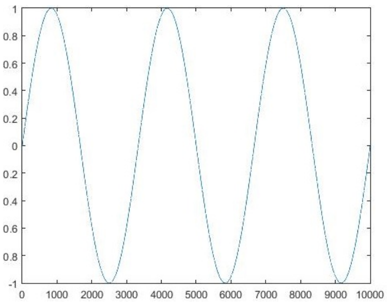

Figure 5.4: *Plot of* $\sin(2\pi 3t)$.

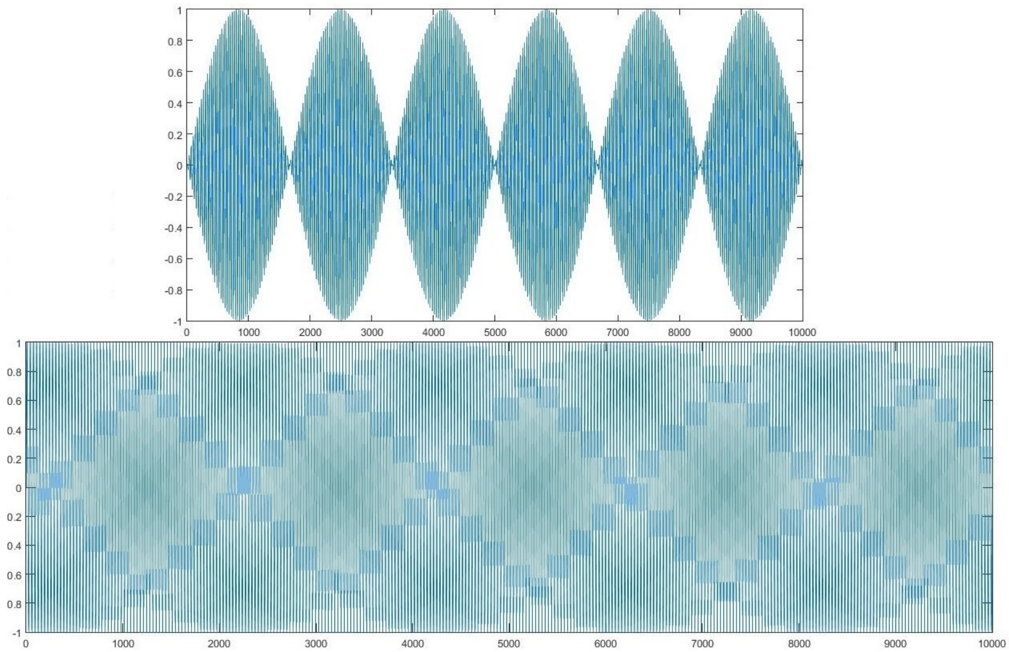

Figure 5.5: *Top: Double-sideband modulation of* $f(t)$, *Bottom: Single-sideband modulation of* $f(t)$.

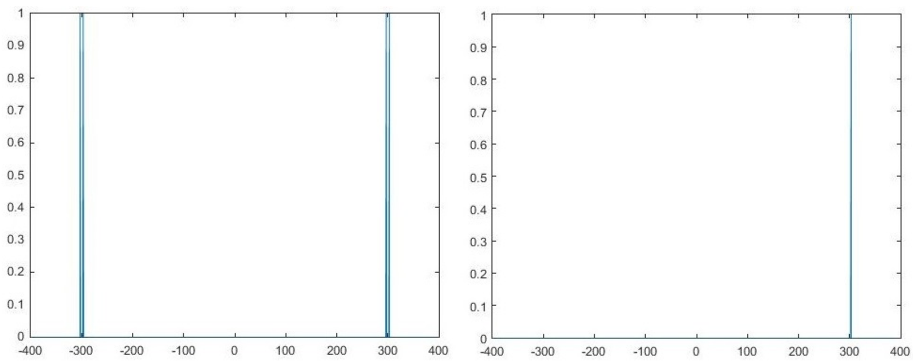

Figure 5.6: *On the left we have the DFT of the doubled-banded modulated signal* $f(t)$. *On the right the DFT of the single-sideband modulated analytic signal* $z(t)$.

As we can see from Figure 5.6(right) that's not the final plot we need. The final DFT plot of the SSB-modulated signal can be obtained by evaluating the real part of $z(t)$ and plotting its Fourier transform, as we did before. Anyway Figure 5.6(right) shows us exactly the halved bandwidth improvements in the SSB modulation. The C++ implementation is useful when it comes to deal with very complicated signals. In those cases we can take a signal $g(t)$, which is a voice recording, and obtain Figures 5.7 and 5.8 which are not so clear anymore but still giving us the same identical signal in the demodulation operation.

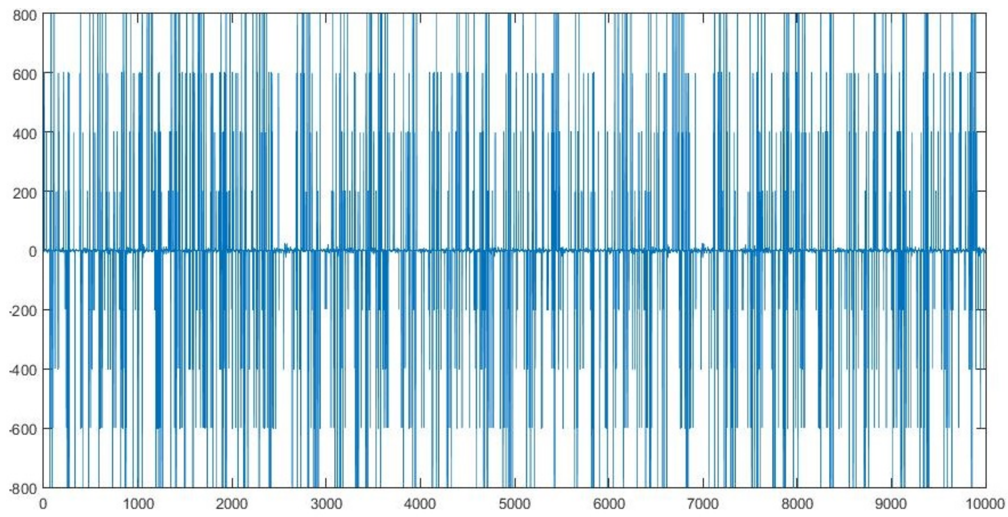

Figure 5.7: *plot of* $g(t)$.

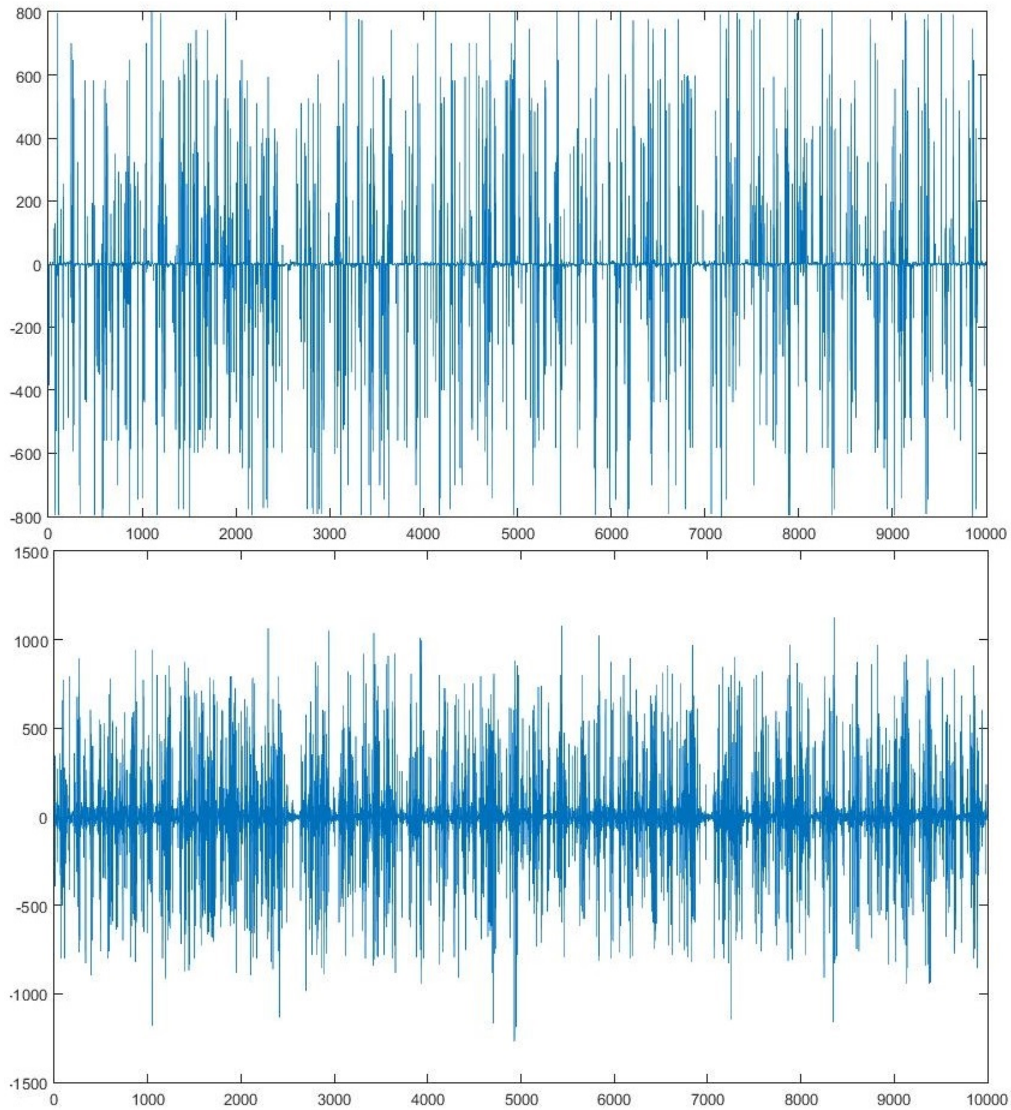

Figure 5.8: *Top: Double-sideband modulation of* $g(t)$, *Bottom: Single-sideband modulation of* $g(t)$.

5.3 Comments

Although there are different methods to modulate a signal (a frequency modulation, or phase modulation) this method is still used in several applications (especially radio communications) and it's the first which is always introduced in basic digital communication courses. Here the key role is played, again, by the analytic expansion of a signal and from this method several improved cases are studied, like the reduced or the suppressed single sideband modulation, where the carrier frequency is as low as possible to improve power consumption and efficiency in transmission.

Chapter 6

The Huang-Hilbert transform

6.1 Introduction

The last implementation we want to present is a tool used in very bad time series (collection of fluctuating variables sampled sequentially in time): nonstationary and nonlinear processes. Nonstationarity means for example a change in time of the mean, while nonlinearity is given by the complexity of the curve generated by the time series. For those circumstances, Fourier transform is not an ideal tool. What is used is the so called **Huang-Hilbert transform (HHT)**. One has to say that this is not a proper transformation; this is more like an empirical algorithm developed at Nasa by E. Huang. It's called Huang-Hilbert because it uses the Hilbert transform in the process and we have to say this method actually lack of theoretical background and a mathematical framework. We can easily separate the entire HHT algorithm into two steps: the **Empirical Mode Decomposition (EMD)** and the **Hilbert's spectrum** analysis. The idea is to take the original time series and decompose it through the EMD into the so called **Intrinsic Mode Functions (IMFs)** and then use them to analyze the signal into a 3-dimensional plot called Hilbert spectrum. Let's start defining what an IMF is:

An **intrinsic mode function (IMF)** is a continuous real-valued function that satisfies two conditions:

(i) The number of local extrema and zero crossings must be equal or differ at most by one.

(ii) The mean value of the two envelopes curves formed by the extrema (local minima/maxima respectively) should be zero at any time.

With envelope we means a smooth curve connecting the points (usually what is used

is a cubic-spline interpolation). We usually have to deal with time series for example in finance to represent the behaviour of certain quantity over time (like the prices of some stocks) and these are just discrete representations of a continuous time series. So call $X(t)$ a continuous time series, the **EMD** algorithm works like this:
Take the time series, for example [1]:

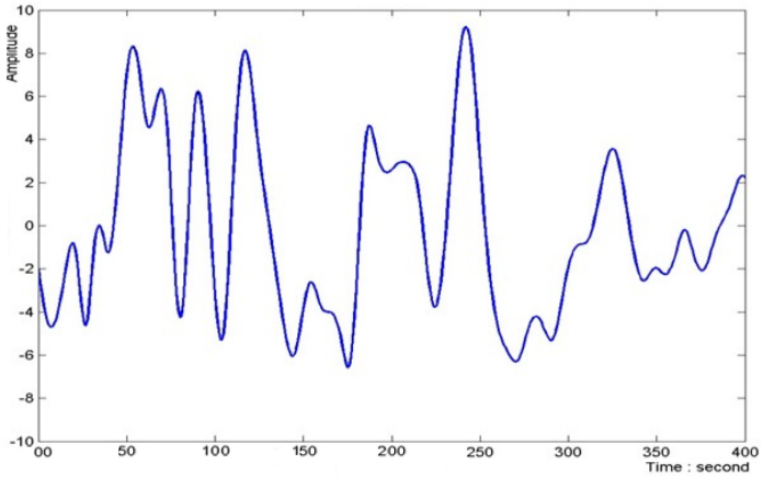

Figure 6.1: *A general time series.*

1) Call $h_{10}(t) = X(t)$, locate each local maximum/minimum of $X(t)$ and construct the two envelope curves by cubic spline interpolation, and represent the mean $m_1(t)$ between the two.

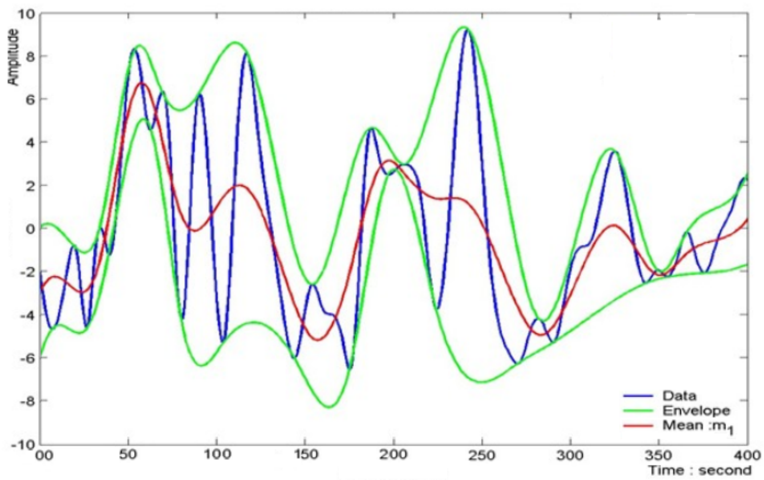

Figure 6.2: *First envelopes and mean.*

2) Calculate $h_{11}(t) = h_{10}(t) - m_1(t) = X(t) - m_1(t)$

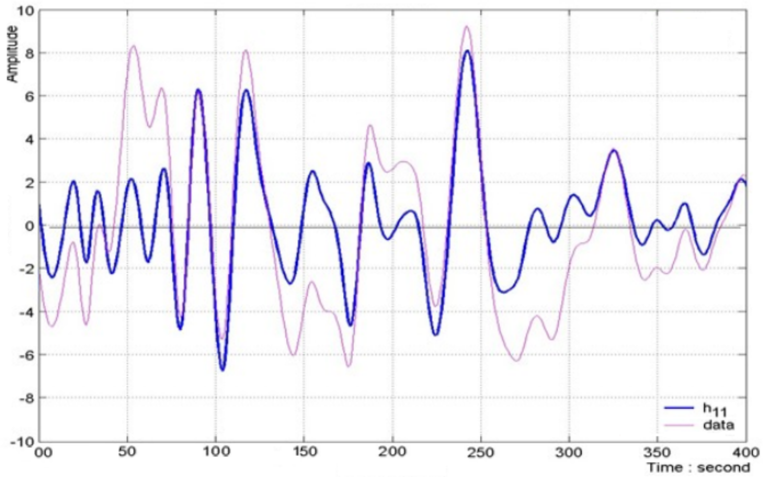

Figure 6.3: *Substracting the previous mean of envelopes from original data.*

3) Use $h_{11}(t)$ as input data and repeat steps 1) and 2) obtaining $h_{12}(t) = h_{11}(t) - m_2(t)$

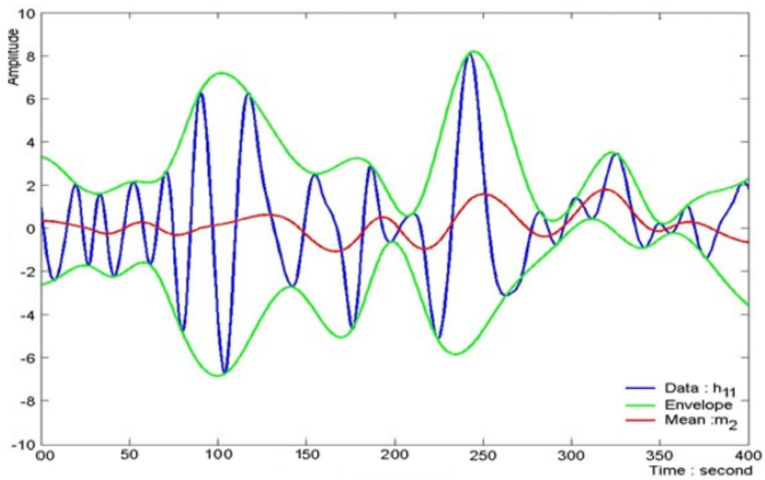

Figure 6.4: *Second envelopes and mean.*

4) Repeat the entire process k times with an appropriate stopping criterion and end-

ing up having $h_{1k}(t) = h_{1(k-1)}(t) - m_k(t)$. So for example iterating three more times the algorithm we have the last iteration obtaining h_{15} (down)

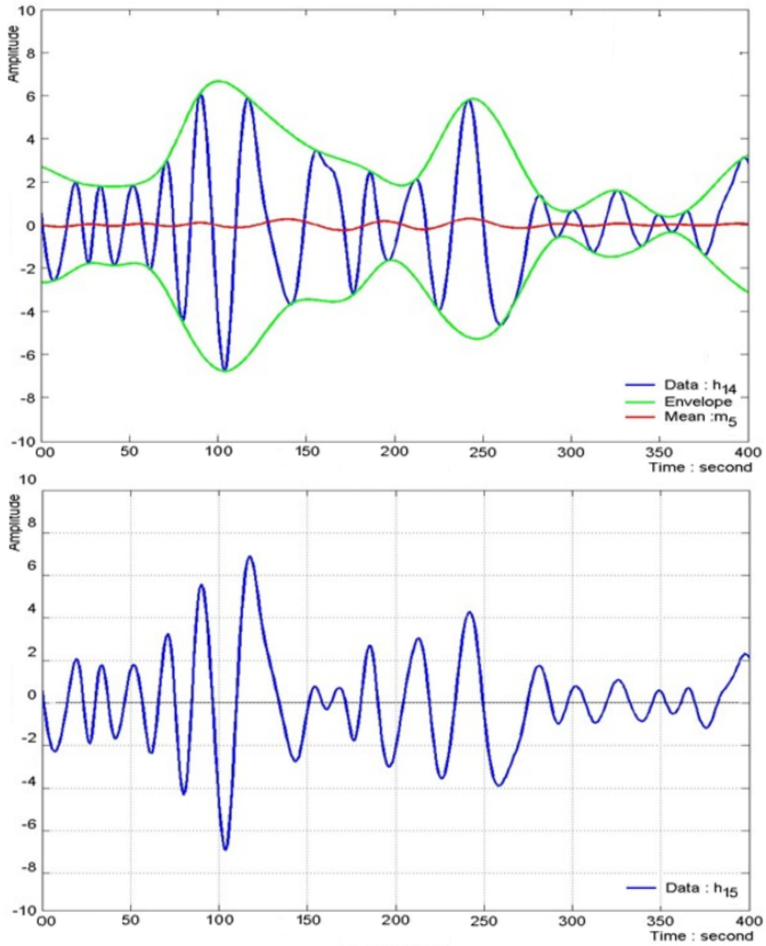

Figure 6.5: *Iterating the process.*

5) Define $h_{20}(t) = X(t) - h_{1k}(t)$ then restart until $h_{nk}(t)$ is monotone or constant for some n.

6) Call $c_i(t) = h_{ik}(t)$ for $i = 1, 2, .., n-1$ and $r_n(t) = h_{nk}$.

The process of extracting the IMFs is called *sifting* process. As stopping criterion what is often used is the standard deviation between two consecutive iterations. When this is below a certain limit (0.3 or 0.5 are typical values) the iteration is stopped.

$$SD_i = \frac{\Sigma_\tau^T |h_{i(k-1)}(t) - h_{ik}(t)|^2}{\Sigma_\tau^T h_{i(k-1)}^2(t)}$$

Where τ and T are the starting and ending instants of time. Notice that this condition implies to have a mean between the envelopes almost zero everywhere. Obviously in step 5) we have to stop when $h_{nk}(t)$ is monotone or constant because in this case no envelope can be produced and no algorithm can be iterated. $c_i(t)$ are IMFs due to the procedure we used to obtain them and $r_n(t)$ is just the residue of the decomposition of the original time series into IMFs. Notice that the first IMF we extract is the one with the highest number of zero crossings among all IMFs, this means also the highest in frequency. An IMF so represent an intrinsic oscillation behaviour (an harmonic), and the first IMF extracted is the prevalent oscillation. Since we remove all the intrinsic oscillations from the time series what we obtain then, that is $r_n(t)$, is just the global trend of the time series. This opens a completely new paradigm of analysis for nonstationary and nonlinear processes and this is why HHT is a very important tool nowadays. From what we have said we should certainly decompose the time series $X(t)$ as (see [7] pag. 46)

$$X(t) = \sum_{i=1}^{n} c_i(t) + r_n(t)$$

Obviously the most important part of this decomposition is the set of all harmonics, while the residue (even if indicates a general trend) can be excluded from the analysis, and if $r_n(t)$ is constant then it is just an offset of our time series. Recalling what we have said about the analytic expansion of a real-valued signal we may want to write it for $X(t)$ as

$$Z(t) = X(t) + iH(X(t))$$

but this time we want to ignore and delete the residue $r_n(t)$. But we need to motivate this reason. First of all if the residue is constant then its Hilbert transform is 0 but if this is not the case, if $r_n(t)$ is monotone, then including it could overpower the harmonics and should therefore be left out (see [7] pag. 50). Obviously the last equation can be written in polar coordinates

$$Z(t) = \sum_{i=1}^{n} [c_i(t) + iH(c_i(t))] = \sum_{i=1}^{n} a_i(t) e^{i\phi_j(t)}$$

Where $\phi_j(t)$ is the $\arg(c_i(t)+iH(c_i(t)))$ and then calling $\omega = \frac{d\phi}{dt}$ the angular frequency, remembering $\omega(t) = 2\pi k(t)$ where $k(t)$ is the frequency we can then write

$$Z(t) = \sum_{i=1}^{n} a_i(t) e^{i(\phi(0)+\int_0^t 2\pi k_j(s)ds)}$$

And our $X(t)$ is just the real part of it. This relation looks similar to the definition of the Fourier series. But they are different in an important aspect: amplitudes $a_i(t)$ and frequencies this time are not constant like in the Fourier case; they depend on time, that's why one may think to the HHT as a generalization of the Fourier transform. This conclusions make us able to think to a 3D plot of what we have said. Indeed if we think to a plane where the two axis are the time and the frequencies, then we can use the z-coordinate to represent the amplitude and this is what is called a **Hilbert spectrum** and it's usually indicated as $H(k,t)$. It does make sense also to define what is called the **marginal Hilbert spectrum** which is

$$h(k) = \int_0^T H(k,t)dt$$

This marginal spectrum can be compared to the amplitude spectrum given by a whatever Fourier transform. The difference is that if we have a frequency different from zero in the amplitude spectrum of the Fourier transform then that particular frequency was present in the whole period of time the transform has been calculated, while in the marginal Hilbert spectrum then that particular frequency appeared in a period of time which can be just a small part of the entire period of time used, and this exact interval of time is given by the complete Hilbert spectrum.

6.2 An example

Using software in [12] and [13] we show now a practical example of the HHT transform. Let's use a record of human voice singing 3 notes.

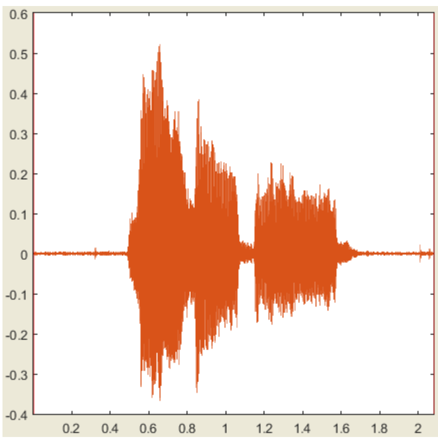

Figure 6.6: *Record of a human voice.*

Apply now the empirical mode decomposition and obtain 16 IMFs.

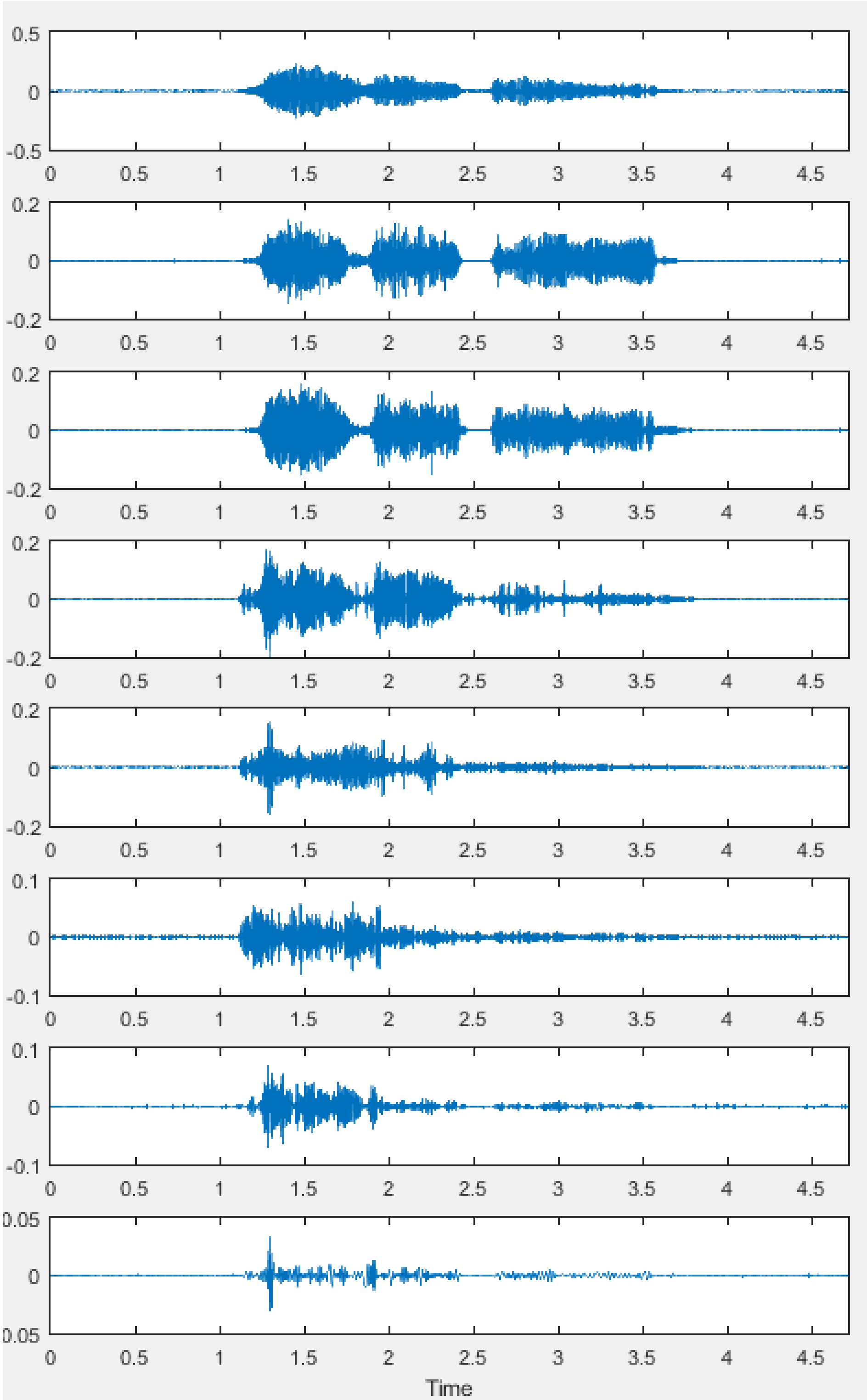

Figure 6.7: *First 8 IMFs.*

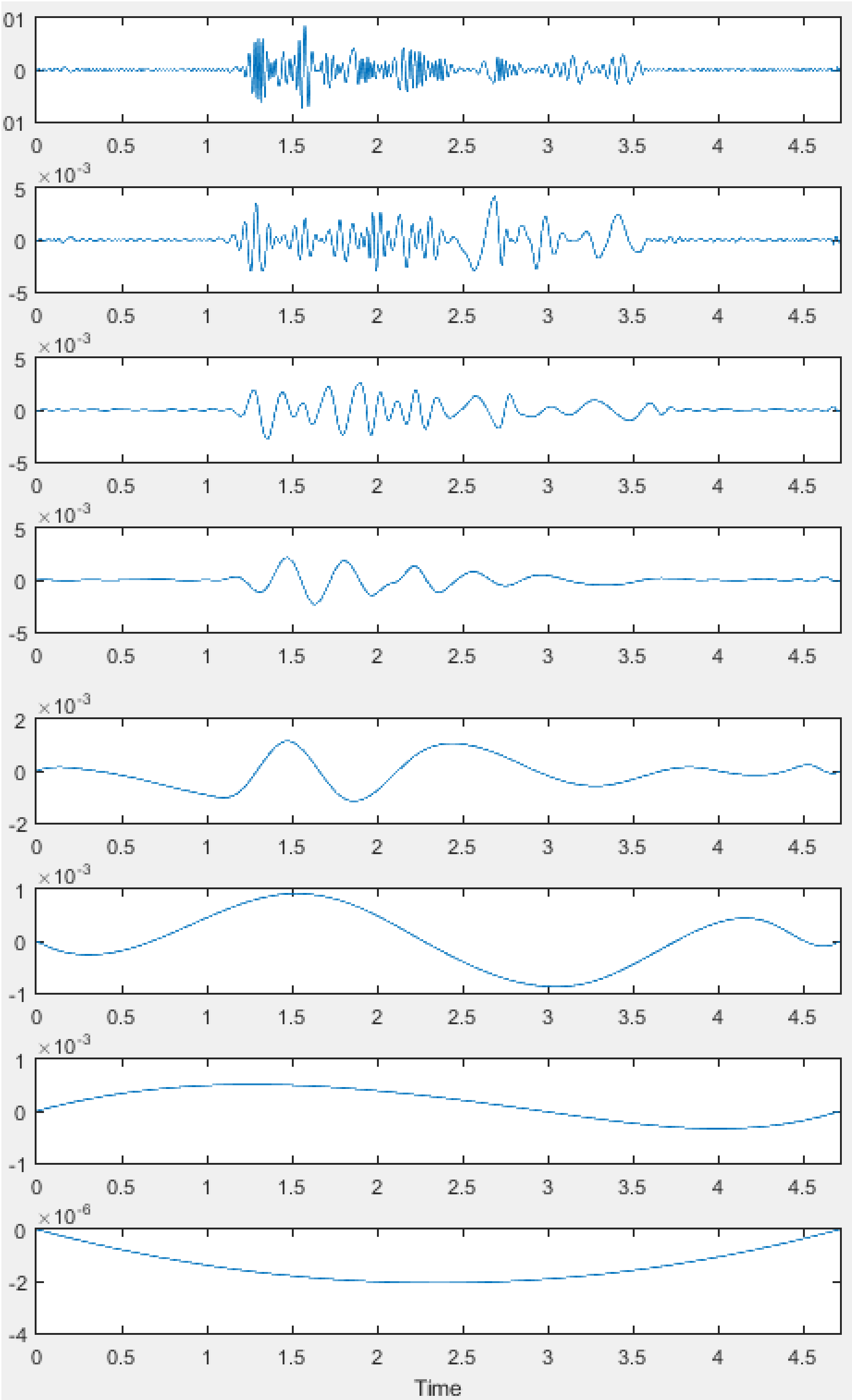

Figure 6.8: *Last 8 IMFs.*

From here we can notice that the 16th diagram is the last IMF we evaluate and it has no zero crossings and only one minima. So these IMFs subtracted from the original signal give us the remaining constant or monotone component. We can notice that each new IMF has less zero crossings than the previous one, and this means in general we are dealing with lower frequencies. It's interesting to notice that this fact of having always less zero crossings for each iteration can be actually conjectured (see [7] pag. 53):

Suppose $X(t)$ is a time series and its Hilbert-Huang transform is a decomposition given by

$$X(t) = \sum_{i=1}^{n} c_i(t) + r_n(t).$$

Then the IMFs, $c_k(t)$, *are decreasing in complexity as k increases. That is, given* $c_k(t)$ and $c_{k+1}(t)$, *then* $c_k(t)$ *has more zero crossings compared to* $c_{k+1}(t)$.

This explains why we can't have a 17th IMF (by definition the difference between zero crossings and local extrema can be at most one).

6.3 Comments

This new way of dealing with signals which are non linear or non stationary, as we said, opens a new paradigm of possible analysis. Nowadays this transform is used in financial, biomedical, physics and engineering field, but is useful also in voice, and iris recognition. Unfortunately there are some problems. As we already said this method is lacking of a mathematical framework and a background theory, and one of the biggest problems is what is called the *end effects*. When we have to deal numerically with this transform it happens that the first and final data samples have only one (instead of two) neighbours data to deal with, this results in a distortion of the envelopes in the sifting process which create an error that propagates itself in all the iterations to extract all the IMFs. The results are IMFs highly distorted at the endpoints. Today there is no complete solution. Another limitation is the lack of resolution this method has in recognising two really adjacent frequencies. To see this just consider two signals:

$$X(t) = \cos(15t) + \cos(t)$$

$$Y(t) = \cos(2t) + \cos(t)$$

As we can notice the $X(t)$ signal has two widely separated frequencies, while $Y(t)$ has two adjacent frequencies. From the decomposition through IMFs we should clearly see the two components of each signal as two different harmonics, so evaluating the first two IMFs c_1^X,c_2^X for $X(t)$ and c_1^Y,c_2^Y for $Y(t)$ we have:

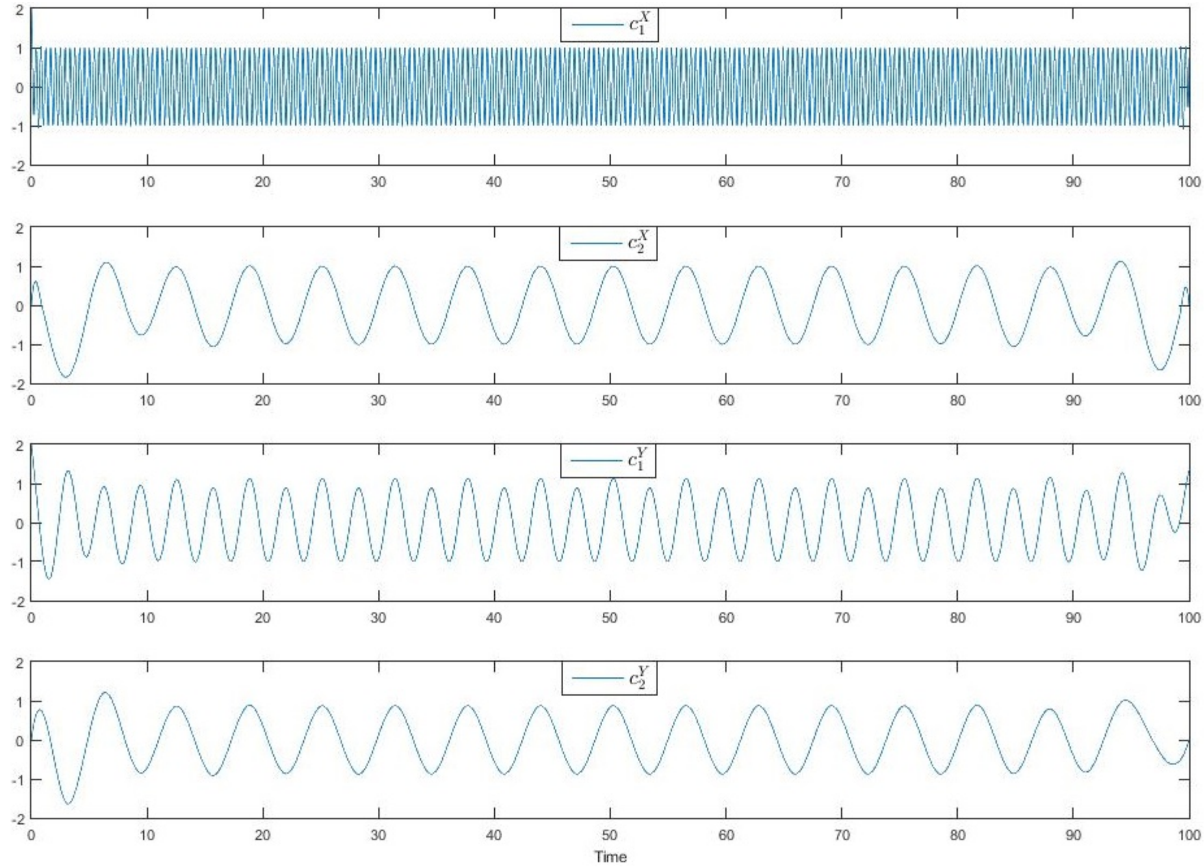

Figure 6.9: *Firs two IMFs of $X(t)$ and $Y(t)$.*

As we said before, the first IMF always reflect the highest oscillatory behaviour so c_1^X and c_1^Y should not differ so much from $\cos(15t)$ and $\cos(2t)$ respectively, and the same argument should hold with the second IMF reflecting a lower oscillatory behaviour that should be, more or less, represented by $\cos(t)$ in both cases. Now the fact is that, as we said, HHT lack of resolution and it's a big problem to deal with adjacent frequencies as in the signal $Y(t)$. This could be easily can be understood taking a look at the next figure

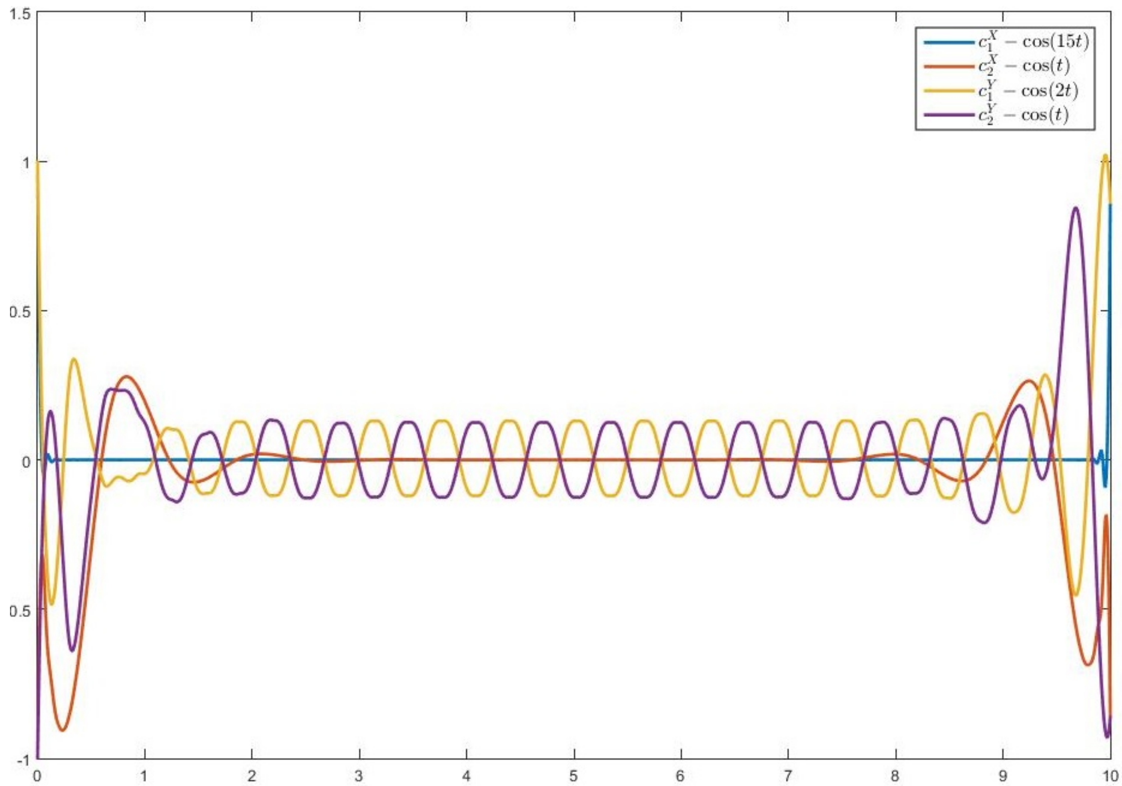

Figure 6.10: *Ignoring edge effects, blue and red lines show a relation with the components of $X(t)$ clearlier than the yellow and purple lines with $Y(t)$.*

Here we clearly see the end effects that propagate in each iteration of the sifting process (in this case not well managed as before by the used software) but, excluding them from our analysis, what we want to highlight here is that while the difference between the first two IMFs of $X(t)$ reflects the two components composing it (blue and red line are zero almost everywhere), the two IMFs of $Y(t)$ have some difference from its two components resulting in a non-zero yellow and purple line. Anyway even if we still have some problems with it, the HHT remains a great discover of the last years, which is improved every year with new techniques.

Appendix A

Proofs of theorems in chapter 1

Proof Theorem 1.1(Cauchy's integral theorem):
If one assumes that the partial derivatives of a holomorphic function are continuous, the Cauchy integral theorem can be proved as a direct consequence of Green's theorem and the fact that the real and imaginary parts of $f = u + iv$ must satisfy the Cauchy-Riemann equations in the region bounded by γ, and moreover in the open neighbourhood R of this region. Cauchy provided this proof, but it was later proved by Goursat without requiring techniques from vector calculus, or the continuity of partial derivatives. We can break the integrand f, as well as the differential dz into their real and imaginary components:

$$f = u + iv$$

$$dz = dx + i\,dy$$

Then

$$\oint_\gamma f(z)\,dz = \oint_\gamma (u + iv)(dx + i\,dy) = \oint_\gamma (u\,dx - v\,dy) + i \oint_\gamma (v\,dx + u\,dy)$$

By Green's theorem, we may then replace the integrals around the closed contour γ with an area integral throughout the domain D that is enclosed by γ as follows:

$$\oint_\gamma (u\,dx - v\,dy) = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$$

$$\oint_\gamma (v\,dx + u\,dy) = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We therefore find that both integrands (and hence their integrals) are zero

$$\iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy = \iint_D \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\, dy = 0$$
$$\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\, dy = 0$$

This gives the desired result

$$\oint_\gamma f(z)\, dz = 0$$

□

Proof theorem 1.2(Jordan's lemma):
Parametrizing $z = Re^{i\phi} = R\cos\phi + iR\sin\phi$ we get $|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax-ay}| = e^{-ay} = e^{-aR\sin\phi}$ and $|dz| = |iRe^{i\phi}d\phi| = Rd\phi$.

$$\int_{C_\epsilon} |e^{iaz}||dz| = \int_0^\pi e^{-aR\sin\phi} Rd\phi = 2\int_0^{\frac{\pi}{2}} e^{-aR\sin\phi} Rd\phi \leq 2\int_0^{\pi/2} e^{-aR2\phi/\pi} Rd\phi$$
$$= -\frac{\pi}{a}[e^{-aR2\phi/\pi}]_{\phi=0}^{\phi=\frac{\pi}{2}} = \frac{\pi}{a}(1 - e^{-aR}) \leq \frac{\pi}{a}$$

The second equality is because $\sin\phi$ is symmetric around $\phi = \pi/2$. Also since $\sin\phi$ is concave in $0 \leq \phi \leq \frac{\pi}{2}$ its graph lies completely above a line connecting its endpoints; therefore we have that $\sin\phi \geq 2\phi/\pi$ when $0 \leq \phi \leq \pi/2$ and hence the first inequality holds. The second inequality holds obviously for $a > 0$.

□

Proof theorem 1.3: Since $f(z)$ has a simple pole at $z = z_0$ we can express its Laurent expansion in some punctured neighbourhood around z_0.

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

Let $g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Then dividing the integral into parts

$$\int_{C_\epsilon} f(z)dz = a_{-1}\int_{C_\epsilon} \frac{1}{z - z_0}dz + \int_{C_\epsilon} g(z)dz \tag{A.1}$$

In the neighbourhood around z_0 the function $g(z)$ is analytical and bounded so $|g(z)| \leq M$ for some constant M. Estimating the integral then

$$|\int_{C_\epsilon} g(z)dz| \leq M(\phi_2 - \phi_1)\epsilon \to 0$$

as $\epsilon \to 0^+$. For the other integral we get, using the parametrization stated in the theorem

$$\int_{C_\epsilon} \frac{1}{z - z_0} dz = \int_{\phi_1}^{\phi_2} \frac{1}{\epsilon e^{i\phi}} \epsilon i e^{i\phi} d\phi = i \int_{\phi_1}^{\phi_2} d\phi = i(\phi_2 - \phi_1)$$

From this and the fact that $a_{-1} =$Res$f(z)$ we get the statement by evaluating

$$\lim_{\epsilon \to 0^+} \int_{C_\epsilon} f(z) d(z) = a_{-1} i(\phi_2 - \phi_1) + 0 = i(\phi_2 - \phi_1) Res_{z=z_0} f(z)$$

□

Appendix B

Fourier inversion and Proofs of theorems in chapter 2

We are going to use several tools, theorems, and corollary to prove the inversion formula of the Fourier transform in $S(\mathbb{R})$. This proofs are taken from [11] (chapter 5) which has more complete and detailed arguments.

The Schwartz space

Recall that the **Schwartz space** on $\mathbb{R}$ consists of the set of all indefinitely differentiable functions f so that f and all its derivatives $f', f'', ..., f^{(l)}, ...,$ are **rapidly decreasing** in the sense that

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty, \text{ for every } k, l \geq 0$$

We denote this space by $S = S(\mathbb{R})$, and is a vector space over $\mathbb{C}$. Moreover, if $f \in S(\mathbb{R})$, we have

$$f'(x) = \frac{df}{dx} \in S(\mathbb{R}), \text{ and } xf(x) \in S(\mathbb{R})$$

This expresses the important fact that the Schwartz space is closed under differentiation and multiplication by polynomials. A simple example of a function in $S(\mathbb{R})$ is the Gaussian defined by

$$f(x) = e^{-x^2}$$

which plays a central role in the theory of the Fourier transform, as well as other fields (for example, probability theory and physics). The derivatives of f are of the form $P(x)e^{-x}$ where P is a polynomial, and this immediately shows that $f \in S(\mathbb{R})$. In fact, e^{-ax^2} belongs to $S(\mathbb{R})$ whenever $a > 0$.

The Fourier transform on $S(\mathbb{R})$

The Fourier transform of a function $f \in S(\mathbb{R})$ is defined by
$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixk}dx.$$
Some simple properties of the Fourier transform are gathered in the following proposition. We use the notation
$$f(x) \to \hat{f}(k)$$
to mean that $\hat{f}$ denotes the Fourier transform of f.

Proposition B1: *If $f \in S(\mathbb{R})$ then:*

(i) $f(x+h) \to \hat{f}(k)e^{2\pi ihk}$ *whenever* $h \in \mathbb{R}$.
(ii) $f(x)e^{-2\pi ixh} \to \hat{f}(k+h)$ *whenever* $h \in \mathbb{R}$.
(iii) $f(\delta x) \to \delta^{-1}\hat{f}(\delta^{-1}k)$ *whenever* $\delta > 0$.
(iv) $f'(x) \to 2\pi ik\hat{f}(k)$.
(v) $-2\pi ixf(x) \to \frac{d}{dk}\hat{f}(k)$.

In particular, except for factors of $2\pi i$, the Fourier transform interchanges differentiation and multiplication by x. We shall return to this point later.

Proof: Property (i) is an immediate consequence of the translation invariance of the integral, and property (ii) follows from the definition. Also in (see [11] pag. 143) we can find a proof for (iii). Integrating by parts gives
$$\int_{-N}^{N} f'(x)e^{-2\pi ixk}dx = [f(x)e^{-2\pi ixk}]_{-N}^{N} + 2\pi ik\int_{-N}^{N} f(x)e^{-2\pi ixk}dx$$
so letting N tend to infinity gives (iv). Finally, to prove property (v), we must show that $\hat{f}$ is differentiable and find its derivative. Let $\epsilon > 0$ and consider
$$\frac{\hat{f}(k+h) - \hat{f}(k)}{h} - (\widehat{-2\pi ixf})(k) =$$
$$\int_{-\infty}^{\infty} f(x)e^{-2\pi ixk}[\frac{e^{-2\pi ixh} - 1}{h} + 2\pi ix]dx$$
Since $f(x)$ and $xf(x)$ are of rapid decrease, there exists an integer N so that $\int_{|x|\geq N} |f(x)|dx \leq \epsilon$ and $\int_{|x|\geq N} |x||f(x)|dx \leq \epsilon$. Moreover, for $|x| \leq N$, there exists h_0 so that $|h| < h_0$ implies
$$|\frac{e^{-2\pi ixh} - 1}{h} + 2\pi ix| \leq \frac{\epsilon}{N}.$$

Hence for $|h| < h_0$ we have

$$|\frac{\hat{f}(k+h) - \hat{f}(k)}{h} - (\widehat{-2\pi ix}f)(k)|$$

$$\leq \int_{-N}^{N} |f(x)e^{-2\pi ixk}[\frac{e^{-2\pi ixh} - 1}{h} + 2\pi ix]|dx + C\epsilon$$

$$\leq C'\epsilon.$$

□

Theorem B2: *If $f \in S(\mathbb{R})$, then $\hat{f} \in S(\mathbb{R})$.*

The proof is an easy application of the fact that the Fourier transform interchanges differentiation and multiplication. In fact, note that if $f \in S(\mathbb{R})$, its Fourier transform $\hat{f}$ is bounded; then also, for each pair of non-negative integers ℓ and m, the expression

$$k^m(\frac{d}{dk})^\ell \hat{f}(k)$$

is bounded, since by the last proposition, it is the Fourier transform of

$$\frac{1}{(2\pi i)^m}(\frac{d}{dx})^m[(-2\pi ix)^\ell f(x)].$$

□

The Gaussians as good kernels

We begin by considering the case $a = \pi$ because of the normalization:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1. \tag{B.1}$$

To see why this is true, we use the multiplicative property of the exponential to reduce the calculation to a two-dimensional integral. More precisely, we can argue as follows:

$$(\int_{-\infty}^{\infty} e^{-\pi x^2} dx)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dxdy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2} rdrd\theta$$

$$= \int_0^{\infty} 2\pi re^{-\pi r^2} dr$$

$$= [-e^{-\pi r^2}]_0^{\infty}$$

$$= 1$$

where we have evaluated the two-dimensional integral using polar coordinates. The fundamental property of the Gaussian which is

Theorem B3: *If* $f(x) = e^{-\pi x^2}$, *then* $\hat{f}(k) = f(k)$.

Proof: Define

$$F(k) = \hat{f}(k) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x k} dx,$$

and observe that $F(0) = 1$, by our previous calculation. By property (v) in Proposition B1, and the fact that $f'(x) = -2\pi x f(x)$, we obtain

$$F'(k) = \int_{-\infty}^{\infty} f(x)(-2\pi i x) e^{-2\pi i x k} dx = i \int_{-\infty}^{\infty} f'(x) e^{-2\pi i x k} dx.$$

By (iv) of the same proposition, we find that

$$F'(k) = i(2\pi i k)\hat{f}(k) = -2\pi k F(k) \ .$$

If we define $G(k) = F(k)e^{\pi k^2}$, then from what we have seen above, it follows that $G'(k) = F'(k)e^{\pi k^2} + 2\pi k G(k) = 0$, hence G is constant. Since $F(0) = 1$, we conclude that G is identically equal to 1, therefore $F(k) = e^{-\pi k^2}$, as was to be shown.
□

The scaling properties of the Fourier transform under dilations yield the following important transformation law, which follows from (iii) in Proposition B1 (with δ replaced by $\delta^{-1/2}$).

Corollary B4: *If* $\delta > 0$ *and* $K_\delta(x) = \delta^{-1/2} e^{-\pi x^2/\delta}$, *then* $\hat{K_\delta}(k) = e^{-\pi \delta k^2}$

We have now constructed a family of good kernels on the real line. Indeed, with

$$K_\delta(x) = \delta^{-1/2} e^{-\pi x^2/\delta},$$

we have:

(i) $\int_{-\infty}^{\infty} K_\delta(x) dx = 1.$

(ii) $\int_{-\infty}^{\infty} |K_\delta(x)| dx \leq M.$

(iii) For every $\eta > 0$, we have $\int_{|x|>\eta} |K_\delta(x)| dx \to 0$ as $\delta \to 0$.

To prove (i), we may change variables and use (B.1), or note that the integral equals $\hat{K_\delta}(0)$, which is 1 by Corollary B4. Since $K_\delta \geq 0$, it is clear that property (ii) is also true. Finally we can again change variables to get

$$\int_{|x|>\eta} |K_\delta(x)| dx = \int_{|y|>\eta/\delta^{1/2}} e^{-\pi y^2} dy \to 0$$

as δ tends to 0.

Theorem B5: *If $f \in S(\mathbb{R})$, then*

$$(f * K_\delta)(x) \to f(x) \text{ uniformly in } x \text{ as } \delta \to 0$$

We first introduce the operation of convolution, which is given as follows.
If $f, g \in S(\mathbb{R})$, their convolution is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

For a fixed value of x, the function $f(x-t)g(t)$ is of rapid decrease in t, hence the integral converges.

Proof: First, we claim that f is uniformly continuous on $\mathbb{R}$. Indeed, given $\epsilon > 0$ there exists $R > 0$ so that $|f(x)| < \epsilon/4$ whenever $|x| \geq R$. Moreover, f is continuous, hence uniformly continuous on the compact interval $[-R,\ R]$, and together with the previous observation, we can find $\eta > 0$ so that $|f(x)-f(y)| < \epsilon$ whenever $|x-y| < \eta$. Using the first property of good kernels, we can write

$$(f * K_\delta)(x) - f(x) = \int_{-\infty}^{\infty} K_\delta(t)[f(x-t) - f(x)]dt,$$

and since $K_\delta \geq 0$, we find

$$|(f * K_\delta)(x) - f(x)| \leq \int_{|t|>\eta} + \int_{|t|\leq\eta} K_\delta(t)|f(x-t) - f(x)|dt.$$

The first integral is small by the third property of good kernels, and the fact that f is bounded, while the second integral is also small since f is uniformly continuous and $\displaystyle\int K_\delta = 1$. This concludes the proof of the theorem. $\square$

The Fourier inversion in $S(\mathbb{R})$

The next result is an identity sometimes called the multiplication formula.

Proposition B6: *If $f, g \in S(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)dy.$$

To prove the proposition, we can just use the Fubini's theorem about the order of integration. Indeed if we write explicitly $\hat{g}(x)$ and we take the esponential part together with $f(x)$ we then obtain that equality. The multiplication formula and the fact that the Gaussian is its own Fourier transform lead to a proof of the first major theorem.

Theorem B7: (Fourier inversion)*If* $f \in S(\mathbb{R})$, *then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{2\pi i x k} dk.$$

Proof: We first claim that

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(k) dk.$$

Let $G_\delta(x) = e^{-\pi\delta x^2}$ so that $\widehat{G_\delta}(k) = K_\delta(k)$. By the multiplication formula we get

$$\int_{-\infty}^{\infty} f(x) K_\delta(x) dx = \int_{-\infty}^{\infty} \hat{f}(k) G_\delta(k) dk.$$

Since K_δ is a good kernel, the first integral goes $f(0)$ as δ tends to 0. Since the second integral clearly converges to $\int_{-\infty}^{\infty} \hat{f}(k) dk$ as δ tends to 0, our claim is proved. In general, by the use of the translation property, let $F(y) = f(y + x)$ so that

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(k) dk = \int_{-\infty}^{\infty} \hat{f}(k) e^{2\pi i x k} dk.$$

As the name of theorem suggests, it provides a formula that inverts the Fourier transform; in fact we see that the Fourier transform is its own inverse except for the change of x to $-x$. More precisely, we may define two mappings $\mathcal{F} : S(\mathbb{R}) \to S(\mathbb{R})$ and $\mathcal{F}^* : S(\mathbb{R}) \to S(\mathbb{R})$ by

$$\mathcal{F}(f)(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x k} dx \text{ and } \mathcal{F}^*(g)(x) = \int_{-\infty}^{\infty} g(k) e^{2\pi i x k} dk$$

Thus $\mathcal{F}$ is the Fourier transform, and Theorem B7 guarantees that $\mathcal{F}^* \circ \mathcal{F} = I$ on $S(\mathbb{R})$, where I is the identity mapping. Moreover, since the definitions of $\mathcal{F}$ and $\mathcal{F}^*$ differ only by a sign in the exponential, we see that $\mathcal{F}(f)(y) = \mathcal{F}^*(f)(-y)$, so we also have $\mathcal{F} \circ \mathcal{F}^* = I$. As a consequence, we conclude that $\mathcal{F}^*$ is the inverse of the Fourier transform on $S(\mathbb{R})$, and we get the following result.

Corollary B8: *The Fourier transform is a bijective mapping on the Schwartz space.* □

Now let's move to the proofs for the other theorems in chapter 2. To prove theorem 2.1 we need first to introduce this two results:

Theorem B9 (Holder's inequality): Suppose $f(t) \in L^p(\mathbb{R}), g(t) \in L^q(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{q} = 1$ for $1 \leq p, q \leq \infty$. Then

$$||fg|| \leq ||f||_p ||g||_q \tag{B.2}$$

Proof:
Let's use **Young's inequality** first. Given a and b two non negative real numbers, and p and q positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$,then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Obviously equality hold only if $a^p = b^q$. claim is certainly true if $a = 0$ or $b = 0$. Therefore, assume $a > 0$ and $b > 0$ in the following. Put $t = 1/p$, and $(1 - t) = 1/q$. Then since the logarithm function is strictly concave, $\log(ta^p + (1-t)b^q) \geq t \log(a^p) + (1 - t) \log(b^q) = log(a) + \log(b) = \log(ab)$. This prove the inequality. Now let's use it. if $||f||_p = 0$, then f is zero almost everywhere, and the product fg is zero almost everywhere, hence the left-hand side of Holder's inequality is zero. The same is true if $||g||_q = 0$. Therefore we may assume $||f||_p > 0$ and $||g||_q > 0$ in the following. if $||f||_p = \infty$ or $||g||_q = \infty$, then the right-hand side of Holder's inequality is infinite. Therefore, we may assume that $||f||_p$ and $||g||_q$ are in $(0, \infty)$. If $p = \infty$ and $q = 1$, then $|fg| \leq ||f||_\infty |g|$ almost everywhere and Holder's inequality follows from the monotonicity of the Lebesgue integral. Similarly for $p = 1$ and $q = \infty$. Therefore, we may also assume $p, q \in (1, \infty)$. Dividing f and g by $||f||_p$ and $||g||_q$, respectively, we can assume that $||f||_p = ||g||_q = 1$. We now use Young's inequality:

$$|f(s)|g(s) \leq \frac{|f(s)|^p}{p} + \frac{|g(s)|^q}{q}, s \in S$$

Where S is the domain; integrating both sides gives

$$||fg||_1 \leq \frac{1}{p} + \frac{1}{q} = 1$$

□
And

Theorem B10: if $f(t) \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$ and $g(t) \in L^1(\mathbb{R})$. then the convolution $(f * g)(t)$ is in $L^p(\mathbb{R})$ and

$$||f * g||_p \leq ||f||_p ||g||_1 \tag{B.3}$$

Proof: For the proof we will consider three different cases, namely $p = 1$, $p = \infty$ and $1 \leq p \leq \infty$.

(i)$p=1$

$$||f*g||_1 = ||\int_{\mathbb{R}} f(s)g(t-s)ds|| = \int_{\mathbb{R}} |\int_{\mathbb{R}} f(s)g(t-s)ds|dt \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |f(s)||g(t-s)|dsdt =$$

$$= \int_{\mathbb{R}} |f(s)| \int_{\mathbb{R}} |g(t-s)|dtds = \int_{\mathbb{R}} |f(s)| \cdot ||g||_1 ds = ||g||_1 \int_{\mathbb{R}} |f(s)|ds = ||g||_1||f||_1 =$$
$$= ||f||_1||g||_1$$

(ii)$p=\infty$, remembering $||f||_\infty = \sup_t |f(t)| < \infty$ if $f(t) \in L^\infty(\mathbb{R})$

$$||f*g||_\infty = \sup_t |\int_{\mathbb{R}} f(s)g(t-s)ds| = \sup_t |\int_{\mathbb{R}} f(t-s)g(s)ds| \leq$$

$$\leq \sup_t \int_{\mathbb{R}} |f(t-s)||g(s)|ds = \int_{\mathbb{R}} \sup_t |f(t-s)||g(s)|ds = \int_{\mathbb{R}} ||f||_\infty |g(s)|ds =$$
$$||f||_\infty \int_{\mathbb{R}} |g(s)|ds = ||f||_\infty ||g||_1$$

(iii) $1 < p < \infty$: Let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

$$|(f*g)(t)| = |\int_{\mathbb{R}} f(s)g(t-s)ds| \leq \int_{\mathbb{R}} |f(s)||g(t-s)|ds =$$

$$= \int_{\mathbb{R}} |f(s)||g(t-s)|^{\frac{1}{p}}|g(t-s)|^{\frac{1}{q}}ds \leq \left(\int_{\mathbb{R}} |f(s)|^p|g(t-s)|ds\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g(t-s)|ds\right)^{\frac{1}{q}} =$$
$$= ||_1^{\frac{1}{q}} \left(\int_{\mathbb{R}} |f(s)|^p g(t-s)|ds\right)^{\frac{1}{p}}$$

Where the second inequality is given by Holder's inequality. Using the $L^p(\mathbb{R})$-norm on both sides gives us

$$||f*g||_p \leq ||g||_1^{1/q}(\int_{\mathbb{R}}\int_{\mathbb{R}} |f(s)|^p|g(t-s)|dsdt)^{1/p} =$$

$$= ||g||_1^{1/q} \left(\int_{\mathbb{R}} |f(s)|^p \int_{\mathbb{R}} |g(t-s)|dsdt\right)^{1/p} =$$

$$||g||_1^{1/q} \left(\int_{\mathbb{R}} |f(s)|^p||g||_1 ds\right)^{1/p} = ||g||_1^{1/q}||g||_1^{1/p}(\int_{\mathbb{R}} |f(s)|^p ds)^{1/p} = ||f||_p||g||_1$$

□

Proof theorem 2.1: Define

$$u_{\epsilon,R}(t) = \begin{cases} \frac{1}{\pi t}, & 0 < \epsilon < |t| < R < \infty \\ 0, & \text{otherwise} \end{cases}$$

and let $H_{\epsilon,R}$ be a particular Hilbert transform, namely

$$H_{\epsilon,R}(x(t)) = \frac{1}{\pi}\int_{\epsilon<|t|<R} \frac{x(s)}{t-s}ds = \int_{-\infty}^{\infty} x(s)u_{\epsilon,R}(t-s)ds = (x * u_{\epsilon,R})(t)$$

where $u_{\epsilon,R} \in L^1(\mathbb{R})$ and $x(t) \in L^p(\mathbb{R})$, $1 < p \leq 2$. Then $\|x * u_{\epsilon,R}\|_1 \leq \|x\|_p \|u_{\epsilon,R}\|_1$. This means that $H_{\epsilon,R}(x(t)) \in L^p(\mathbb{R})$, $1 < p \leq 2$ by Theorem B9 and B10 and then we can write

$$F(H_{\epsilon,R}(x(t))) = F((x * u_{\epsilon,R})(t)) = F(x(t)) \cdot F(u_{\epsilon,R}(t))$$

and we need to see that $F(u_{\epsilon,R}(t))$ is

$$\begin{aligned}
F(u_{\epsilon,R}(t)) &= \int_{\epsilon<|t|<R} \frac{e^{-2\pi ikt}}{\pi t}dt = \int_{-R}^{-\epsilon} \frac{e^{-2\pi ikt}}{\pi t}dt + \int_{\epsilon}^{R} \frac{e^{-2\pi ikt}}{\pi t}dt \\
&= -\int_{\epsilon}^{R} \frac{e^{2\pi ikt}}{\pi t}dt + \int_{\epsilon}^{R} \frac{e^{-2\pi ikt}}{\pi t}dt = -\frac{1}{\pi}\int_{\epsilon}^{R} \frac{e^{2\pi ikt} - e^{-2\pi ikt}}{t}dt \\
&= -\frac{2i}{\pi}\int_{\epsilon}^{R} \frac{\sin(2\pi kt)}{t}dt = -\frac{2i\mathrm{sgn}(2\pi k)}{\pi}\int_{2\pi|k|\epsilon}^{2\pi|k|R} \frac{\sin t}{t}dt \\
&= -\frac{2i\mathrm{sgn}(k)}{\pi}\int_{2\pi|k|\epsilon}^{2\pi|k|R} \frac{\sin t}{t}d
\end{aligned}$$

The last integral has the limit $\pi/2$ as $\epsilon \to 0^+, R \to \infty$. This can be shown involving basic techniques with integrals in the complex plane. Obviously, we have that $F(u_{\epsilon,R}(t)) \to -i\mathrm{sgn(k)}$ as $\epsilon \to 0^+$, $R \to \infty$ for every k.

Since $F(u_{\epsilon,R}(t)) \to -i\mathrm{sgn(k)}$ as $\epsilon \to 0^+, R \to \infty$ we have that $|F(u_{\epsilon,R}(t))| \leq C$ for some C depending on the values of ϵ and R. We can now show that

$$H(x(t)) = \lim_{\epsilon\to 0+}\lim_{R\to\infty} H_{\epsilon,R}(x(t)) = F^{-1}(-i\mathrm{sgn}(k) \cdot F(x(t)))$$

by making use of Lebesgue's theorem of dominated convergence. For $1 < p \leq 2$

$$\begin{aligned}
&\lim_{\epsilon\to 0+}\lim_{R\to\infty} \|H_{\epsilon,R}(x(t)) - F^{-1}(-i\mathrm{sgn}(k) \cdot F(x(t)))\|_p \\
&\lim_{\epsilon\to 0+}\lim_{R\to\infty} \|F(H_{\epsilon,R}(x(t))) - (-i\mathrm{sgn}(k) \cdot F(x(t)))\|_p \\
&\lim_{\epsilon\to 0+}\lim_{R\to\infty} \|F(x(t)) \cdot F(u_{\epsilon,R}(t)) - (-i\mathrm{sgn}(k) \cdot F(x(t)))\|_p \\
&\lim_{\epsilon\to 0^+}\lim_{R\to\infty} ||(F(u_{\epsilon,R}(t)) - (-i\mathrm{sgn}(k))) \cdot F(x(t))\|_p = 0
\end{aligned}$$

where the first equality is because of Parseval's identity. Thus we have established the limit above (in the sense of the L^p norm). Now, by just taking the Fourier transform of both sides we get because of the Fourier inversion theorem that

$$F(H(x(t))) = (-i\mathrm{sgn}(k)) \cdot F(x(t))$$

and the proof is complete.

□

Appendix C

C++ code for ECG and AM modulation

To implement the discrete Fourier transform (and its inverse) we use the simplest implementation, without recalling faster algorithms like the fast Fourier transform which is not so intuitive by simply looking at the code

```cpp
void forwardDFT(const double *s, const int &N, double *&a, double *&b)
{
    for (int k = 0; k < N; ++k) {
        a[k] = b[k] = 0;
        for (int x = 0; x < N; ++x) {
            a[k] += s[x] * cos(2 * PI * k * x / N);
            b[k] -= s[x] * sin(2 * PI * k * x / N);
        }
    }
}

void inverseDFT( const double *a, const double *b, const int &N, long double *&s)
{
 for (int x = 0; x < N; ++x) {
   s[x] = a[0];
     for (int k = 1; k < N ; ++k) {
       s[x] += (a[k] * cos(2* PI * k * x / N) ) - (b[k] * sin(2* PI * k * x / N));

        }
    }
}
```

It has to be noticed that in this case the forward discrete Fourier transform only has the real part because the signal is actually real so its imaginary part is 0 always. Regarding the filter it is:

```
void filter(double *M, double *n, double *D, double *d,double *s,double *F,int N)
{
for(int i=0; i<N; i++)
{
//FIND ALL MAX's and put them into M and n vectors
if((s[i]>s[i+1])&&(s[i]>s[i-1])){if(s[i]>0){M[i]=s[i];} else {n[i]=s[i];}}
if((s[i]==s[i+1])&&(s[i+1]>s[i+2])){if(s[i]>0){M[i]=s[i];} else {n[i]=s[i];}}
/* Cut out all values below a certain limit(=what we consider noise by looking at
the signal) and open a window for each peak above the limit and construct the
filtered signal F */
if(M[i]>0.3){for(int j=0; j<20; j++){F[i-j+10]=s[i-j+10];F[i+j]=s[i+j];}}
}
}
```

And finally the Hilbert transform

```
void hilbert( double *m, double *l, const int &c, long double *&hil_out,const
double *a, const double *b)
{for(int i=1; i<(c/2); i++){m[i]=b[i]*(2);l[i]=-a[i]*2;}
for(int i=(c/2)+1; i<c; i++){m[i]=0;l[i]=0;}
inverseDFT(m,l,c,hil_out);}
```

C++ code for amplitude modulation

In order to compare the doubled-side band modulation with the SSB, we have implemented both in C++. Then the doubled bandwidth modulations is given by

```
//AM-MODULATION (doubled bandwith)
void am_modulation(int N, double *O, double *s, double f, double t, double T)
{
for(int i=0; i<N; i++){O[i]=(s[i]*cos(2*PI*f*t)); t+=T;}
t=0;
}
```

And the SSB modulations by

```
//SINGLE SIDE-BAND MODULATION
void single_sideband_modulation(int N, double *p, double *o, double *s,
double f,double t, double T,double *&a, double *&b, long double *hil_out)
{
 /* CALCULATING FIRST DFT IN ORDER TO BE ABLE TO CALCULATE ALSO HILBERT TRANSFORM
 AND USE IT IN MODULATION */
 int const c=N;
 double *m = new double [c]; //Auxiliar variable used in hilbert transform
 double *l = new double [c]; //Auxiliar variable used in hilbert transform
 forwardDFT(s,N,a,b);
 hilbert(m,l,c,hil_out,a,b);
 for(int i=0; i<N; i++){a[i]=0; b[i]=0;} //Reset of a,b vectors
//SSB-MODULATION
 for(int i=0; i<N; i++){o[i]=((s[i]*cos(2*PI*f*t))-(hil_out[i]*sin(2*PI*f*t)));
 p[i]=((s[i]*sin(2*PI*f*t))+(hil_out[i]*cos(2*PI*f*t))); t+=T;}
t=0;
}
```

Since we are writing code to be implemented in digital analysis the demodulations are given by

```
void AMdemodulation(double *m,int N,double *o,double F, double t,double T)
{
for(int i=0; i<N; i++){m[i]=((2*(o[i]*cos(2*PI*F*t)))/(1+cos(4*PI*F*t))); t+=T;}
}
```

And

```
void SSBdemodulation(double *m,int N,double *o,double F,double t, double T,
long double *hil_out)
{
for(int i=0; i<N; i++)
{m[i]=(((2*(o[i]*cos(2*PI*F*t)))+(hil_out[i]*sin(4*PI*F*t)))/(1+cos(4*PI*F*t)));
t+=T;}
}
```

Bibliography

[1] B. Barnett - *Tutorial on the empirical mode decomposition* "http://slideplayer.com/slide/5294982/", 2015.

[2] A. Goldberger, L. Amaral - *PhysioBank, PhysioToolkit, and PhysioNet: Components of a New Research Resource for Complex Physiologic Signals* - Circulation Electronic Pages: "http://circ.ahajournals.org/content/101/23/e215.full", "http://www.physionet.com", 2000.

[3] N. E. Huang, Z. Shen, S. R. Long, M. C. Wu, H. H. Shih, Q. Zheng, N. Yen, C. C. Tung, H. H. Liu - *The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis* - Proc. Roy. Soc. Lond., 1998.

[4] J. Hunter, B. Nachtergaele - *Applied Analysis* - Department of Mathematics (University of California at Davis), 2000.

[5] A. Husin - *The Hilbert Transform* - Department of Mathematics (Uppsala University), 2013.

[6] M. Johansson - *The Hilbert Transform* - Department of Mathematics (Vaxjo University).

[7] M. Klingspor - *Hilbert transform: Mathematical theory and applications to signal processing* - Mathematics institute (University of Linköpings), 2015.

[8] S. S. Kohli, N. Makwana, N. Mishra, B. Sagar - *Hilbert Transform Based Adaptive ECG R-Peak Detection Technique* - International Journal of Electrical and Computer Engineering (IJECE), Vol. 2, No. 5, 2012 pp. 639-643.

[9] M. Reed, B. Simon - *Methods of modern mathematical physics* - Academic Press, inc., 1970.

[10] W. Rudin - *Real and complex analysis - Third edition* - McGraw-Hill Book Company, 1987.

[11] E. M. Stein, R. Shakarchi - *Fourier Analysis - An introduction* - Princeton University Press, (2003).

[12] A. Tan - *Hilbert-Huang Transform* "https://es.mathworks.com/matlabcentral/fileexchange/19681-hilbert-huang-transform", 2016.

[13] A. Tan - *Speech Snipper* "http://es.mathworks.com/matlabcentral/fileexchange/11002-speech-snipper?requestedDomain=www.mathworks.com", 2016.

[14] Z. Wang - *Hilbert transform applications in signal analysis and non-parametric identification of linear and nonlinear systems* - Missouri University of Science and Technology, (2011).